A TREATISE

ON

ALGEBRA.

VOL. II.

ON SYMBOLICAL ALGEBRA,

AND ITS

APPLICATIONS

TO THE

GEOMETRY OF POSITION.

BY

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I. M.
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PREFACE.

I HAVE endeavoured, in the present volume, to present the principles and applications of Symbolical, in immediate sequence to those of Arithmetical, Algebra, and at the same time to preserve that strict logical order and simplicity of form and statement which is essential to an elementary work. This is a task of no ordinary difficulty, more particularly when the great generality of the language of Symbolical Algebra and the wide range of its applications are considered; and this difficulty has not been a little increased, in the present instance, by the wide departure of my own views of its principles from those which have been commonly entertained.

It is true that the same views of the relations of the principles of Arithmetical and Symbolical Algebra formed the basis of my first publication on Algebra in 1830: but not only was the nature of the dependance of Symbolical upon Arithmetical Algebra very imperfectly developed in that work, but no sufficient attempt was made to reduce its principles and their applications to a complete and regular system, all whose parts were connected with each other: they have consequently been sometimes controverted upon grounds more or less erroneous; and notwithstanding a very general acknowledgment of their theoretical authority, they have hitherto exercised very little influence upon the views of elementary writers on Algebra.

It may likewise be very reasonably contended that the reduction of such principles, as those which I have ventured

to put forward, to an elementary form, in which they may be fully understood by an ordinary student, is the only practical and decisive test, I will not say of their correctness, but of their value: for we are very apt to conclude that the most difficult theories and researches which have become familiar to us from long study and contemplation, may be made equally clear and intelligible to others as well as to ourselves: and though I will not say that I feel perfectly secure that I may not have been, in some degree, under the influence of this very common source of self-deception and error, yet I have adopted the only course which was open to me, in order to bring this question to an issue, by embodying my own views in an elementary work, and by suppressing as much as possible any original or other researches, which might be considered likely to interfere with its complete and systematic developement.

It is from the relations of space that Symbolical Algebra derives its largest range of interpretations, as well as the chief sources of its power of dealing with those. branches of science and natural philosophy which are essentially connected with them: it is for this reason that I have endeavoured to associate Algebra with Geometry thoughout the whole course of its developement, beginning with the geometrical interpretation of the signs + and − when considered with reference to each other, and advancing to that of the various other signs which are symbolized by the roots of 1: we are thus enabled, in the present volume, to bring the Geometry of Position, embracing the whole theory of lines considered both in relative position and magnitude and the properties of rectilineal figures, under the dominion of Algebra: in a subsequent volume this application will be further extended to the Geometry of Situation, (where lines are considered in

their absolute as well as in their relative position with respect to each other), and also to the theory of curves.

The theory of the roots of 1 is so important, not merely with reference to the signs of affection which they symbolize, but likewise in the exposition of the general theory of equations and in all the higher branches of Symbolical Algebra, that I have thought it expedient to give it with unusual fulness and detail: such roots may be considered as forming the connecting link between Arithmetical and Symbolical Algebra, without whose aid the two sciences could be very imperfectly separated from each other.

I have not entered further into the general theory of equations than was necessary to enable me to exhibit the theory of their general solution, as far it can be carried by existing methods, reserving the more complete exposition of their properties and of the methods employed for their numerical solution, to a subsequent volume.

The plan which I have adopted necessarily brings Trigonometry, or to speak more properly, Goniometry, within the compass of the present volume, not merely as forming the most essential element in the application of Algebra to the Geometry of Position, but as intimately connected with many important analytical theories.

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ERRATA.

Page	Line	From	Error	Correction
87	9	bottom	$a+\frac{(1-\sqrt{5})\,a}{2}$	$x=a+\frac{(1-\sqrt{5})\,a}{2}$.
—	10	bottom	$a+\frac{(1+\sqrt{5})\,a}{2}$	$x=a+\frac{(1+\sqrt{5})\,a}{2}$.
108	15, 17	top	$(-1)^{\frac{1}{3}}$	$(1)^{\frac{1}{3}}$.
197	11	bottom	$\cos\theta$	$\cos-\theta$.
206	7	top	$-2\sqrt[6]{(a^2+b^2)}\cos\left(\frac{2\pi-\theta}{3}\right)$	$2\sqrt[6]{(a^2+b^2)}\cos\left(\frac{2\pi-\theta}{3}\right)$.
—	8	bottom	$-2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi-\theta}{3}\right)$	$2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi-\theta}{3}\right)$.
289	3	bottom	$\log\left(1-\frac{b}{a}e^{C\sqrt{-1}}\right)$	$\log\left(1-\frac{b}{a}e^{-C\sqrt{-1}}\right)$.
344	4	top	$2\cos\frac{2\pi-\theta}{2}$	$2\cos\left(\frac{2\pi-\theta}{3}\right)$.
357	11	bottom	$=\frac{R}{2}$	$=r$.

It is feared that very few of the errors have been noticed, no sufficient time having been devoted to the revision of the work.

CHAPTER XI.

ON THE OPERATIONS OF ADDITION AND SUBTRACTION IN SYMBOLICAL ALGEBRA.

543. THE symbols in Arithmetical Algebra represent numbers, whether abstract or concrete, whole or fractional, and the operations to which they are subject are assumed to be identical in meaning and extent with the operations of the same name in common Arithmetic: the only distinction between the two sciences consisting in the substitution of general symbols for digital numbers.

Distinction between Arithmetic and Arithmetical Algebra.

Thus, if a be added to b, as in the expression $a+b$, it is assumed that a and b are either abstract numbers or concrete numbers of the same kind: if b be subtracted from a, as in the expression $a-b$, it is assumed that a is greater than b, which implies likewise that they are numbers of the same kind: if a be multiplied by b, as in the expression ab (Art. 34), or if a be divided by b, as in the expression $\frac{a}{b}$ (Art. 71), it is assumed that b is an abstract number. In all these cases, the operation required to be performed, whether it be addition or subtraction, multiplication or division, is clearly defined and understood, and the result which is obtained, is a necessary consequence of the definition: the same observation applies to all the results of Arithmetical Algebra.

544. But the symbols, which are thus employed, do not convey, either to the eye or to the mind, in the same manner as digital numbers and geometrical lines, the limitations of value to which they are subject in Arithmetical Algebra: for they are equally competent to represent quantities of all kinds, and of all relations of magnitude. But if we venture to ascribe to them a perfect generality of value, (upon which a conventional limitation was imposed in Arithmetical Algebra), it will be found to involve, as an immediate and necessary consequence, the

The assumption of the unlimited values of the symbols employed involves the necessary recognition of the independent use of the signs + and −.

recognition of the use of symbols preceded by the signs + and −, without any direct reference to their connection with other symbols.

Thus, in the expression $a - b$, if we are authorized to assume a to be either *greater* or *less* than b, we may replace a by the equivalent expression $b + c$ in one case, and by $b - c$ in the other: in the first case, we get $a - b = b + c - b = b - b + c$ (Art. 22), $= 0 + c$ (Art. 16) $= + c = c$: and in the second $a - b = b - c - b$ $= b - b - c = 0 - c = - c$. The first result is recognized in Arithmetical Algebra (Art. 23): *but there is no result in Arithmetical Algebra which corresponds to the second:* inasmuch as it is assumed that no operation can be performed and therefore no result can be obtained, when a is less than b, in the expression $a - b$ (Art. 13).*

Positive and negative quantities.

545. Symbols, preceded by the signs + or −, without any connection with other symbols, are called *positive* and *negative* (Art. 32) symbols, or *positive* and *negative* quantities: such symbols are also said to be *affected* with the signs + and −. *Positive* symbols and the numbers which they represent, form the subjects of the operations both of Arithmetical and Symbolical Algebra: but *negative* symbols, whatever be the nature of the quantities which the *unaffected* symbols represent, belong exclusively to the province of Symbolical Algebra.

Assumptions made in symbolical addition and subtraction.

546. The following are the assumptions, upon which the rules of operation in Symbolical Addition and Subtraction are founded.

1st. Symbols, which are general in form, are equally general in representation and value.

2nd. The rules of the operations of addition and subtraction in Arithmetical Algebra, when applied to symbols which are general in form though restricted in value, are applied, without alteration, in Symbolical Algebra, where the symbols are general in their value as well as in their form.

It will follow from this second assumption, as will be afterwards more fully shewn, that all the results of the operations of addition and subtraction in Arithmetical Algebra, will be results likewise of Symbolical Algebra, but not conversely.

* If we assume symbols to be capable of all values, from *zero* upwards, we may likewise include *zero* in their number: upon this assumption, the expressions $a + b$ and $a - b$ will become $0 + b$ and $0 - b$, or $+ b$ and $- b$, or b and $- b$ respectively, when a becomes equal to *zero*: this is another mode of deriving the conclusion in the text.

547. Proceeding upon the assumptions made in the last Article, the rule for Symbolical Addition may be stated as follows:

Rule for symbolical addition.

"Write all the *addends* or *summands* (Art. 24, Ex. 1.) in the same line, preceded by their proper signs, collecting *like* terms (Art. 28) into one (Art. 29): and arrange the terms of the result or *sum* in any order, whether alphabetical or not, which may be considered most symmetrical or most convenient."

It will be understood that *negative* (Art. 545) as well as *positive* symbols or expressions may be the subjects of this operation, and it is therefore not necessary, as in Arithmetical Algebra, that the first term of the final result should be positive (Art. 22).

Negative terms may occupy the first place in the results of Symbolical Algebra.

548. The following are examples of Symbolical Addition.

Examples.

(1) Add together $3a$ and $5a$.

$$3a+5a=8a \quad \text{(Art. 29)}.$$

(2) Add together $3a$ and $-5a$.

$$3a-5a=-2a \quad \text{(Art. 31)}.$$

This is exclusively a result of Symbolical Algebra.

(3) Add together $-3a$ and $5a$.

$$-3a+5a=2a.$$

This result, which is obtained by the Rule, is equivalent to that which would arise from the subtraction of $3a$ from $5a$: or, in other words, the addition of $-3a$ to $5a$ in Symbolical Algebra, is equivalent to the subtraction of $3a$ from $5a$ in Arithmetical Algebra.

(4) Add together $-3a$ and $-5a$.

$$-3a-5a=-8a \quad \text{(Art. 31)}.$$

This is exclusively a result of Symbolical Algebra: in contrasting it however with Ex. 1, it merely differs from it in the use of the sign $-$ throughout, instead of the sign $+$.

(5)		(6)		(7)	
	$3a$		$3x^2$		$-abc$
	$-5a$		$-\ x^2$		$12abc$
	$+7a$		$-7x^2$		$13abc$
	$-4a$		$-4x^2$		$-20abc$
	a		$-9x^2$		$4abc$

In these examples, the coefficients of the like terms, which have the same sign, are added together, and to the difference of the sums, preceded by the sign of the greater, is subjoined the symbolical part of the several like terms: it is the rule given in Art. 31, applied without any reference to the signs of the first term.

$$(8)\quad \begin{array}{r} 3a-4b \\ -7a+8b \\ a-3b \\ \hline -3a+\ b \end{array} \qquad (9)\quad \begin{array}{r} -7x^2+6xy-\ 7y^2 \\ 8x^2-4xy-\ \ y^2 \\ -3x^2-\ \ xy+10y^2 \\ \hline -2x^2+\ \ xy+\ 2y^2 \end{array}$$

In these examples, the sets of like terms are severally combined into one (Art. 31), and arranged, in the result, in alphabetical order, no regard being paid to the placing a positive term, when any exists, in the first place.

$$(10)\quad \begin{array}{r} -3a-4b+5c \\ -\ \ a+2b-3d \\ 3b-4c+6e \\ 7c-8d-9e \\ \hline -4a+\ \ b+8c-11d-3e \end{array}$$

The several sets of like terms are collected together out of the several addends.

$$(11)\quad \begin{array}{r} 7x^2-\ \ 4ax+\ \ \ a^2 \\ -10x^2-11ax-\ \ 4a^2 \\ -\ \ 3x^2+13ax+10a^2 \\ \hline -\ \ 6x^2-\ \ 2ax+\ \ 7a^2 \end{array}$$

The alphabetical order of the symbols is, in this case, reversed. It should be kept in mind in this and in all other cases that the arrangement of the terms in the final result, does not affect its value or signification, but is merely adopted as an aid to the eye or to the memory, or with reference to peculiar circumstances connected with some one or more of the symbols involved: see the Examples in Art. 33.

Rule for Subtraction in Symbolical Algebra.

549. The rule for subtraction in Symbolical Algebra is derived, by virtue of the assumptions in Art. 546, from the corresponding rule in Arithmetical Algebra: it may be stated as follows.

"To the minuend or minuends, add (Art. 547) the several terms of the subtrahend or subtrahends with their signs changed from + to − and from − to +."

The following are examples of Symbolical subtraction. Examples.

(1) From a subtract $-b$.

The result is $a+b$: or the symbolical *difference* of a and $-b$ is equivalent to the *sum* of a and b.

$$\begin{array}{lr@{\qquad}lr@{\qquad}lr@{\qquad}lr}
(2) & 7a & (3) & -7a & (4) & 7a & (5) & -7a \\
 & 3a & & -3a & & -3a & & 3a \\
\cline{2-2}\cline{4-4}\cline{6-6}\cline{8-8}
 & 4a & & -4a & & 10a & & -10a
\end{array}$$

In these examples, the minuend and subtrahend are written underneath each other, as in common Arithmetic: the results are severally the same as in the following examples of addition. (Art. 547)

$$\begin{array}{r@{\qquad}r@{\qquad}r@{\qquad}r}
7a & -7a & 7a & -7a \\
-3a & 3a & 3a & -3a \\
\hline
\end{array}$$

$$\begin{array}{lr@{\qquad}lr@{\qquad}lr@{\qquad}lr}
(6) & a+b & (7) & a-b & (8) & a+b & (9) & -a-b \\
 & a-b & & a+b & & -a+b & & a-b \\
\cline{2-2}\cline{4-4}\cline{6-6}\cline{8-8}
 & 2b & & -2b & & 2a & & -2a
\end{array}$$

These examples are respectively equivalent to

(6) $a+b-(a-b)$, or $a+b-a+b=2b$.

(7) $a-b-(a+b)$, or $a-b-a-b=-2b$.

(8) $a+b-(-a+b)$, or $a+b+a-b=2a$.

(9) $-a-b-(a-b)$, or $-a-b-a+b=-2a$.

The terms of the several subtrahends are included between brackets, and, when the brackets are removed, all their signs are changed (Art. 24).

$$\begin{array}{lr}
(10) & a^3+3a^2x+3ax^2+x^3 \\
 & a^3-3a^2x+3ax^2-x^3 \\
\cline{2-2}
 & 6a^2x+2x^3
\end{array}$$

$$\begin{array}{lrrrrr}
(11) & 3a- & 4b+ & 7c- & 9d & \\
 & & 2b- & 10c- & 6d+ & 14e \\
\cline{2-6}
 & 3a- & 6b+ & 17c- & 3d- & 14e
\end{array}$$

(12) From $3x-7y$ subtract $x+2y$ and $-7x+4y$.

$$3x-7y-(x+2y)-(-7x+4y)$$
$$=3x-7y-x-2y+7x-4y=9x-13y.$$

The several subtrahends included between brackets and preceded by the sign $-$, are written in the same line with the minuend; the brackets are subsequently removed and the signs of the several terms which they include changed, in conformity with the Rule.

$$\begin{aligned}(13)\quad & x^2+2xy-y^2-\{x^2+xy-y^2+(2xy-x^2-y^2)\}\\ &=x^2+2xy-y^2-x^2-xy+y^2-(2xy-x^2-y^2)\\ &= xy-2xy+x^2+y^2\\ &= x^2-xy+y^2.\end{aligned}$$

In this case, we first remove the exterior brackets and reduce to their most simple equivalent form the terms which are external to those which remain: we then remove the remaining brackets and arrange the terms, when reduced, in alphabetical order (Arts. 20 and 21.)

$$\begin{aligned}(14)\quad & a-[a+b-\{a+b+c-(a+b+c+d)\}]\\ &=a-a-b+\{a+b+c-(a+b+c+d)\}\\ &= -b+a+b+c-(a+b+c+d)\\ &= a+c-a-b-c-d\\ &= -b-d, \text{ or } -(b+d).\end{aligned}$$

In this case, we have a triple set of brackets which are successively removed, and the like terms, which they involve, are obliterated or reduced into one. See the Examples in Art. 33.

The same terms are used in Arithmetical and Symbolical Algebra. In Arithmetical Algebra, the definitions determine the rules of operation: in Symbolical Algebra the rules of operation determine

550. In the exposition and exemplification of the preceding rules, we have felt it to be unnecessary to repeat definitions and assumptions which are common to Symbolical and Arithmetical Algebra: such are the ordinary uses and meanings of the signs $+$ and $-$, of coefficients (Art. 25), of like and unlike terms (Art. 28), of indices and powers (Arts. 38 and 39), and the methods of denoting the ordinary operations (Art. 9).

551. The use however, of the same terms in these two sciences will by no means imply that they possess the same meaning in all their applications. In Arithmetic and Arithmetical Algebra, addition and subtraction are defined or understood in their ordinary sense, and the rules of operation are deduced from the definitions: in Symbolical Algebra, we adopt the rules of operation

which are thence derived, extending their application to all values of the symbols and adopting also as the subject matter of our operations or of our reasonings, whatever quantities or forms of symbolical expression may result from this extension: but, *inasmuch as in many cases, the operations required to be performed are impossible, and their results inexplicable, in their ordinary sense, it follows that the meaning of the operations performed, as well as of the results obtained under such circumstances, must be derived from the assumed rules, and not from their definitions or assumed meanings, as in Arithmetical Algebra.* the meaning of the operations themselves, and of their results.

We will endeavour to illustrate this important and fundamental distinction between these two sciences in the case of the two operations which form the subject of this Chapter.

552. The rule of subtraction, derived rigorously from the definition or assumed meaning of that operation in Arithmetical Algebra, directs us to change the signs of the terms of the subtrahend and to write them, when so changed, in the same line with the terms of the minuend, incorporating, by a proper rule, like terms into one. In the application of this rule, three cases will present themselves, which it may be proper to consider separately. Consideration of three different cases of subtraction in the transition from Arithmetical to Symbolical Algebra.

1st. Where the *subtrahend* is obviously less than the *minuend*. When the subtrahend is less than the minuend. Examples.

If the minuend be $a+c$ and the subtrahend be a, then $a+c-a=+c=c$. If the minuend be $3a+7b$ and the subtrahend be $2a+6b$, then $3a+7b-(2a+6b)=3a+7b-2a-6b=a+b$.

In these examples, the results follow necessarily from the definition of subtraction in its ordinary usage, merely supposing that the symbols represent magnitudes of the same kind.

2nd. Where the *subtrahend* may or may not be less than the *minuend*, according to the relation of the values of the symbols involved. When the subtrahend may be less than the minuend, but not necessarily so. Examples.

If the minuend be $3a+4b$ and the subtrahend $2a+5b$, then $3a+4b-(2a+5b)=3a+4b-2a-5b=a-b$.

If a be greater than b, this is a result of Arithmetical Algebra, but it ceases to be so when a is less than b: as long therefore as the relation of the values of a and b remains undetermined, it is uncertain whether it is a result of Arithmetical or of Symbolical Algebra: it is one of an infinite number of cases in which these two sciences may be said to inosculate with each other.

If the minuend be $3a-4b$ and the subtrahend be $2a-b$, then $3a-4b-(2a-b)=3a-4b-2a+b=a-3b$. Unless a be greater than $3b$ this is an example of Symbolical Algebra only.

When the subtrahend is greater than the minuend. Examples.

3rd. Where the *subtrahend* is obviously greater than the *minuend.*

If the minuend be a and the subtrahend $a+c$, then

$$a-(a+c)=a-a-c=-c.$$

If the minuend be $2a+3b$ and the subtrahend $3a+4b$: then

$$\begin{aligned} &2a+3b-(3a+4b) \\ &=2a+3b-3a-4b=-a-b=-(a+b). \end{aligned}$$

If the minuend be $2a+3b+4c$ and the subtrahend be $4a+7b+10c$: then

$$\begin{aligned} &2a+3b+4c-(4a+7b+10c) \\ &=2a+3b+4c-4a-7b-10c \\ &=-2a-4b-6c=-(2a+4b+6c). \end{aligned}$$

In these examples, the results are obtained, by the application of the rule for the removal of the brackets when preceded by the sign *minus* and for the incorporation of like terms, under circumstances which are not recognized in Arithmetical Algebra, inasmuch as they are not necessary consequences of the definition of the operation of subtraction, though they are necessary results of the unlimited application of the rule for performing it.

In Symbolical Algebra, it is indifferent in what order the terms succeed each other.

553. Again, in Arithmetical Algebra, it has been shewn to be indifferent in what order, terms connected by the signs *plus* and *minus*, succeed each other, so long as a positive term occupies the first place (Art. 22) and the several operations indicated are possible*. Thus $a+b$ is equivalent to $b+a$: $a-b+c$ is equivalent to $a+c-b$ or $c+a-b$ or $c-b+a$*. The same rule is transferred to Symbolical Algebra, without any restriction with respect to the values or signs of the symbols involved, in virtue of the assumptions made in Art. 546. Thus

* Art. 21 and note. Thus if b was greater than a, but less than $a+c$, the operation, or rather succession of operations, indicated in the expression $a-b+c$ would be impossible, but would cease to be so in the expression $a+c-b$: it is however convenient, even in Arithmetical Algebra, to consider an expression as representing any value which an interchange of its terms would render it capable of expressing: in virtue of this convention, we might consider $-b+a$, as equivalent to $a-b$, and recognise its use in Arithmetical Algebra, if a was greater than b.

$a-b$ is equivalent, in this latter science, to $-b+a$, for all values of a and b: $-3a+5a$ is equivalent to $5a-3a$: $a-b+c$ is equivalent to $a+c-b$ or $c+a-b$ or $c-b+a$ or $-b+a+c$ or $-b+c+a$, and similarly in all other cases.

554. Inasmuch as the results of symbolical addition and subtraction are obtained from an assumed rule of operation, and not from the definition of the operation itself, it will follow that their meaning, when capable of being interpreted, must be dependent upon the conditions which they are required to satisfy: but as the rules for performing these operations and the results obtained are or may be made* identical in those two sciences in all cases which are equally within their province, it is allowable to assume that the operations and their results, within those limits, possess precisely the same meaning: it is only when the results of these rules are not common to Arithmetical Algebra, that it will be found necessary to resort to an interpretation of their meaning, upon principles which we shall proceed to establish and to exemplify in the case of the operations which are under consideration.

The meaning of those results of symbolical algebra, which are not common to arithmetical algebra, must be ascertained by interpretation.

555. The addition of a symbol preceded by a negative sign is equivalent to the subtraction of the same symbol preceded by a positive sign and conversely.

Conditions for the interpretation of positive and negative symbols.

Thus $$a+(-b)=a-b=a-(+b):$$
$$a-(-b)=a+b=a+(+b)\dagger.$$

It appears, therefore, that in the case of negative symbols, the operation of addition is no longer associated with the fundamental idea of *increase*, nor that of subtraction with that of *decrease*: and thus a change of sign from *plus* to *minus*, in the symbol operated upon, is equivalent to a change of operation from *addition* to *subtraction* and conversely.

556. The signs *plus* and *minus*, when prefixed to symbols denoting quantities of the same kind, cannot denote modifica-

The signs + and − used independently, can symbolize convertible affections of magnitude only.

* Thus if a be greater than b, the symbolical result $-b+a$ is convertible into $a-b$, which is, under this form, a result of arithmetical algebra.

† These formulæ express the rule for the concurrence of *like* and *unlike* signs in symbolical addition and subtraction, which is as follows: "when two *unlike* signs come together, they are replaced by the single sign −; and when two *like* signs come together, they are replaced by the single sign +."

tions of magnitude*, but only such affections or qualities of the magnitudes represented, as are convertible by the operations of addition and subtraction: it is on this account that $-a$ can admit of no interpretation, as compared with a or $+a$, when a denotes an abstract number, to which no qualities are attributed.

Possible and impossible quantities.

557. Quantities and their symbols are said to be *real* or *possible*, when they can be shewn to correspond to real or possible existences: in all other cases, they are said to be *unreal, impossible* or *imaginary*. It will follow, therefore, that when positive symbols represent real quantities, the same symbols with a negative sign will be said to be impossible or imaginary, whenever they are not capable of an interpretation, which is consistent with the conditions they are required to satisfy. It remains to shew that there exist large classes of magnitudes which possess qualities which can be correctly symbolized by the signs + and −, and that consequently the terms *negative* and *impossible*† are not coextensive in their application.

Interpretation of the signs + and − in the case of symbols representing lines in geometry.

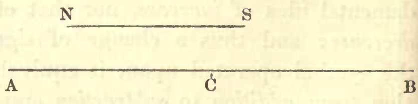
558. Our first example of the existence of qualities of magnitudes which can be thus symbolized will be in expressing the opposite directions of lines in geometry, and which will be found to constitute one of the most extensive applications of Symbolical Algebra: the discussion of the following problem will form the most simple introduction to this most important theory.

Problem.

"A traveller moves southwards for a miles, and then returns northwards for b miles: what is his final distance from the point of departure?"

Its solution by the principles of Arithmetical Algebra.

Let A be the point of departure and AB the distance, expressed in magnitude by the symbol a, to



which he travels southwards: let BC be the distance, expressed in magnitude by the symbol b, through which he returns north-

* For if $+a$ and $-b$ continue to denote magnitudes of the same kind, they may be replaced by the ordinary symbols of Arithmetical Algebra, such as c and d, when $c+d=a+(-b)=a-b$ is always greater than $c-d$ or $a-(-b)$ or $a+b$, results which are contradictory to each other.

† So numerous are the cases in which *negative* quantities admit of a consistent interpretation, that the term *impossible* has never been applied to them: it has been uniformly applied however to a second class of symbolical quantities, though not, as will be hereafter shewn, with perfect propriety.

wards: his final distance AC, when he stops, from the point of departure, is the excess of AB above AC, and is correctly represented by $a-b$: or if we suppose c to represent the distance AC, then we have

$$a-b=c.$$

As long as AB is greater than BC, or a is greater than b, the traveller continues on the same side of the point of departure A, and the solution of the problem is strictly within the limits of Arithmetical Algebra: but if we suppose the traveller to return farther northwards than he went, in the first instance, southwards, or AB to be less than BC, then his distance

AC to the north of the point of departure A, is not capable of being represented in Arithmetical Algebra by $a-b$, since a is less than b, and the operation thus indicated is impossible: but inasmuch as, in this case, AC the final distance of the traveller to the north of A is the excess of BC above AB, or of b above a, it will be correctly represented in Arithmetical Algebra by $b-a$: and if $b=a+c$, we shall have

$$b-a=c.$$

Two distinct cases of this problem when solved by the principles of Arithmetical Algebra.

There are therefore two distinct cases of this problem, when solved by the principles of Arithmetical Algebra, according as the traveller stops on the *south* or on the *north* of the point of departure: and it will be observed that the solution obtained in each case expresses the *absolute magnitude* of the final distance only, and not its *quality* or *affection*, whether *south* or *north*.

Its geometrical solution.

In the geometrical solution of the problem, the distances are expressed and exhibited to the eye, both in quality and magnitude: and there is no such interruption of continuity in passing from *south* to *north* of the point of departure, or, in other words, through the *zero* point, as occurs in the solution of the problem by the principles of Arithmetical Algebra.

Its solution by the principles of Symbolical Algebra.

Let us next consider the solution of this problem by the principles of Symbolical Algebra.

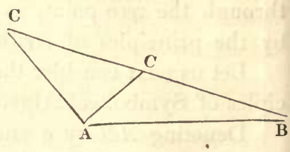
Denoting AB by a and BC by b, when AB is greater than BC, the distance AC from the point of departure is denoted by $a-b$: and inasmuch as the operation denoted by $-$ is assumed to be possible for all relations of value of a and b (Art. 546), we may suppose b equal to $a+c$ as well as to $a-c$, or to be greater as well as less than a: in one case we get, when $b=a-c$,

$a-b=a-(a-c)=+c=c$, and in the other, when $b=a+c$, $a-b=a-(a+c)=-c$.

Assuming the first result $+c$ or c, which is common to Arithmetical Algebra, to represent a distance to the *south* of the point of departure, the second or $-c$, which belongs to Symbolical Algebra alone, will represent a corresponding distance to the *north* of the point of departure: for the expression $a-b$ becomes $+c$ or $-c$, according as b is replaced by $a-c$, or by $a+c$, or according as the traveller is on the *south* or the *north* of the point of departure, and his actual distance considered with respect to magnitude only, is, in both cases, correctly expressed by c: it follows, therefore, that the signs + and − applied to the distance c, under such circumstances, when considered with reference to each other, will symbolize its *qualities* whether *south* or *north*: thus if $+c$ denotes a distance to the *south*, $-c$ will denote an equal distance to the *north* and conversely; such an interpretation of the meaning of these signs when thus applied, is consistent with the assumption that the operation denoted by − in the expression $a-b$ is equally possible for all values of the symbols: and it also enables us to include under one formula, and therefore under one solution, the two cases of the problem which are required to be separately considered in Arithmetical Algebra*.

The signs + and − used as signs of affection of symbols denoting lines, symbolize opposite directions, and opposite directions only.

559. We are enabled to conclude, from the discussion of the preceding problem, that the signs + and −, when thus used *independently*, symbolize relative qualities of the specific magnitude which the symbol c expresses, where the nature of the relation between them is determined by the conditions which they are required to satisfy: thus in the case which we have been considering, it is *opposition of direction, and opposition of direction alone* which these signs can correctly express: for if we suppose AB (a) to be the distance to which the traveller proceeds southwards, and BC or b the distance to which he proceeds not northwards, but in any other direction, such as north north east, then his final distance AC from the point of departure will not be expressed by $a-b$, whatever be the relation of the values of a and b: in other words, if we replace b by $a-c$ or by $a+c$, the result $+c$ or c in one case and $-c$ in



* See appendix.

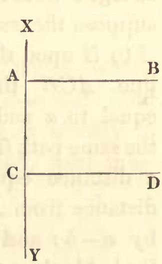
the other, will not express the final distances of the traveller either in magnitude or direction, and consequently will admit of no interpretation which is relative to the problem proposed*.

560. The same problem may be variously modified in form, without altering the essential conditions upon which its solution by the principles of Symbolical Algebra depends: thus instead of estimating distances *northwards* and *southwards* of a point of departure, they may be estimated, with reference to the observer's position, as lines to the *right* or *left*, *up* and *down*, *to* and *from*, or as reckoned upon any *straight line intermediate in position to these three fundamental directions* †: and in all these cases the quantity and direction of the movement *to* or *from* the same point, or the *magnitude* and *direction* of the geometrical lines which express them, will be correctly symbolized by the signs + and −: the steps of the processes by which these conclusions are deduced, are absolutely identical with those which have been followed in the discussion of the problem in Art. 558.

The same problem may be variously modified in form and lead to the same conclusion.

561. Equal lines which have the same *relative* though different *absolute* positions, will be expressed by the same symbol with the same sign: for if AB and CD be taken

equal to each other upon the same line, and be estimated, with reference to their mode of description, in the same direction, they are identical in magnitude and direction, and are therefore expressed by the same symbol which symbolizes the first, and by the same sign which symbolizes the second. Again, if two equal lines AB and CD be drawn perpendicular to the same line or axis XY, and their directions estimated from it, AB and CD being identical in magnitude and direction, will be expressed by the same symbol with the same sign: and, more generally, equal *parallel* lines drawn or estimated in the same direction, whether with reference to a line or plane, will be likewise represented by the same symbol with

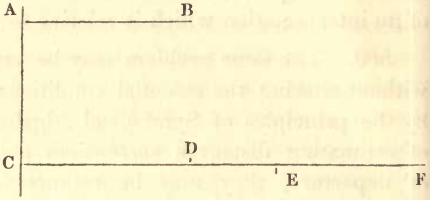


Equal lines which have the same relative but different absolute positions are expressed by the same symbol with the same sign.

* For the *angle between the directions* of the movements, a relation different from that of mere opposition is involved, as well as the *magnitudes* of a and b, in the determination of this distance and its direction.

† If we conceive three planes to pass through the given points at right angles to each other, two of which also pass through the eye of the observer, the three fundamental directions in question will be estimated upon their common intersections.

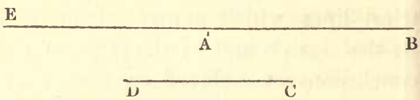
the same sign: for if AB be identical in magnitude and in direction with CD, and if EF be identical in magnitude and direction with CD, it will follow that AB and EF, which may represent any two equal and parallel lines drawn towards the same



parts, are identical in magnitude and direction, and therefore represented by the same symbol with the same sign.

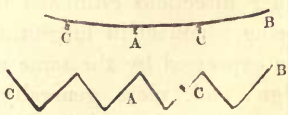
Equal parallel lines estimated in opposite directions will be expressed by the same symbol with different signs, one + and the other −.

562. It will follow, therefore, that if AB and CD be two equal parallel lines drawn in opposite directions, they will be expressed by the same symbol with different signs, one *plus* and the other *minus*: for if BA be produced to E, making AE equal to AB or CD: then CD and AE are identical in magnitude and direction, and therefore represented by the same symbol with the same sign: if AB therefore be expressed by a, AE and therefore CD will be expressed by $-a$ and conversely.



The same signs may express opposite directions not in a straight line.

563. The same system of symbolization will equally express opposite directions of movement upon curved or upon straight lines: for, if in the problem proposed in Art. 558 we suppose the traveller to move from A to B upon the curved or zigzag line ACB through a distance equal to a and then return upon the same path from B to C through a distance equal to b, his final



distance from A, *reckoned upon the same path*, will be expressed by $a-b$: and as the conditions of the problem and of Symbolical Algebra will admit of every possible relation of value of a and b, we may equally replace b by $a-c$ or by $a+c$: and the results $+c$ or c in one case and $-c$ in the other, will express the final distance of the traveller from A: every step of the reasoning by which this conclusion is deduced being the same as in the problem referred to.

564. Whatever conclusions have been deduced in the preceding articles respecting the symbolization of the affections and magnitudes of lines, will be equally applicable to those magnitudes whose affections and magnitudes can be symbolized or expressed by the lines themselves. Thus forces in opposite directions, such as a force which pushes and a force which pulls, or a force which attracts and a force which repels, may be expressed by lines in opposite directions whose lengths are proportional to the magnitudes of the forces: and it consequently follows that the same forces can be expressed by symbols affected by the signs + and −. Again, time past and time to come and whatever magnitudes are capable of continuous and indefinite extension, and therefore of a geometrical representation, are likewise susceptible of affections which can be similarly symbolized.

The same principle of symbolization is applicable to magnitudes which can be expressed in quantity and affection by lines.

565. We shall conclude our observations upon this subject with the discussion of one more example of a problem of very extensive application.

Symbolization of property possessed and owed.

A merchant possesses a pounds and owes b pounds: his substance is therefore $a-b$, when a is greater than b.

But since a and b may possess every relation of value, we may replace b by $a-c$ or by $a+c$, according as a is greater or less than b: in the first case we get

$$a-b=a-(a-c)=c:$$

and in the second,

$$a-b=a-(a+c)=-c:$$

if c therefore express his substance or property, when *solvent*, $-c$ will express the amount of his debts when *insolvent:* and if from the use of + and − as signs of affection or quality in this case, we pass to their use as signs of operation, then inasmuch as (Art. 555)

$$a+(-c)=a-c \text{ and } a-(-c)=a+c,$$

it will follow, that the *addition* of a debt $(-c)$ is equivalent to the *subtraction* of property c of an equal amount, and the *subtraction* of a debt $(-c)$ is equivalent to the *addition* of property (c) of an equal amount: and it consequently appears that the *subtraction* of a debt, in the language of Symbolical Algebra, is not its *obliteration* or *removal*, but the change of its affection or

character, from money or property owed to money or property possessed.

The preceding examples of the interpretation of the meaning of negative quantities, and of the operations to which they are subjected, will be sufficient to shew the student that the province of Symbolical Algebra is not unreal and imaginary, but that it comprehends the representation of large classes of real existences, including some of the most important of those which are the objects of mathematical and philosophical reasoning.

CHAPTER XII.

ON THE OPERATIONS OF MULTIPLICATION AND DIVISION IN SYMBOLICAL ALGEBRA.

566. THE fundamental assumptions which were made in Art. 546, with respect to Symbolical addition and subtraction, are equally applicable, and for the same reasons, to Symbolical multiplication and division: they are as follows: Fundamental assumptions in symbolical multiplication and division.

1st. Symbols which are general in form, are equally general in representation and value.

2nd. The rules of the operations of multiplication and division in Arithmetical Algebra, when applied to symbols which are general in form though restricted in value, are applied without alteration to the operations bearing the same names in Symbolical Algebra, when the symbols are general in their value as well as in their form.

It will follow from the second assumption that all the results of the operations of multiplication and division in Arithmetical Algebra, will be results likewise of Symbolical Algebra, but not conversely.

567. The same three Cases of the operation of multiplication present themselves in Symbolical and in Arithmetical Algebra: they are as follows: Three cases of symbolical multiplication.

1st. When the multiplicand and multiplier are mononomials.

2nd. When the multiplicand is a polynomial and the multiplier a mononomial.

3rd. When both the multiplicand and multiplier are polynomials.

568. In Arithmetical Algebra, the rule for the *concurrence* of *like* and *unlike* signs (Art. 57,) is required in the 2nd and 3rd Cases only: but in Symbolical Algebra, the occurrence of symbols or single terms affected with the signs + and − used *independently* (Art. 544), renders its application necessary in all the three Cases under consideration. The rule for the concurrence of like and unlike signs required in all the three cases.

569. In order to shew that the Rule of signs is a necessary consequence of the assumptions made in Art. 566, we shall con- Its deduction from the assumptions in Art. 566.

sider the product of $a-b$ and $c-d$ as determined by the principles of Arithmetical Algebra (Art. 56), which is

$$(a-b)(c-d)=ab-ad-bc+bd. \qquad (1).$$

Assuming, therefore, the *permanence* of this result, or in other words, the equivalence of the two members of which it is composed, *for all values of the symbols,* we may suppose two of their number to become successively equal to *zero:* thus, if we suppose $b=0$ and $d=0$, the product (1) in question becomes

1st. $(a-0)(c-0)=ac-a\times 0-0\times c+0\times 0$, or $a\times b=ab$, obliterating the terms which involve zero.

If we suppose $b=0$ and $c=0$, we get,

2nd. $(a-0)(0-d)=a\times 0-ad-0\times 0+0\times d$, or $a\times -d=-ad$.

If we suppose $a=0$ and $d=0$, we get,

3rd. $(0-b)(c-0)=0\times c-0\times 0-bc+b\times 0$, or $-b\times c=-bc$.

If we suppose $a=0$ and $c=0$, we get,

4th. $(0-b)(0-d)=0\times 0-0\times d-b\times 0+bd$, or $-b\times -d=bd$.

It follows therefore generally, as a necessary consequence of the assumptions (Art. 566), which form the foundation of the

The Rule. results of multiplication in Symbolical Algebra, that "*when two like signs, whether + and + or − and −, concur in multiplication, they are replaced in the product by the single sign +: and that when two unlike signs similarly concur, whether + and −, or − and +, they are replaced in the product by the single sign −.*"

Rule for symbolical multiplication for Case 1.

570. We now proceed to exemplify, in their order, the three different Cases of Symbolical multiplication.

Case 1. When the multiplicand and multiplier are mononomials.

Rule. "In finding the mononomial product we must determine first, its sign; secondly, its coefficient; and lastly, its literal part."

"Its sign is found by the rule for the concurrence of like and unlike signs which is deduced in the last article: when this sign is +, it is commonly suppressed."

"Its coefficient is the product of the coefficients of the several factors (Art. 37): if the coefficient be 1, (Art. 30), it is generally suppressed, as not necessary to be exhibited."

"Its literal part is found by writing the several letters and their powers in immediate succession after each other, incorporating powers of the same letter into one by the rule given in Art. 41."

It is proved in Arithmetical Algebra, (Art. 37), and therefore assumed in Symbolical Algebra, (Art. 566), that it is indifferent in what order the several component factors of a product succeed each other: it is on this account that it is usual to follow the alphabetical order, whenever the peculiar circumstances of the question under consideration do not render a departure from it convenient.

571. The following are examples: Examples of Case 1.

(1) $a \times b = ab$ or $+a \times +b = +ab$.

(2) $-a \times -b = ab$ or $-a \times -b = +ab$.

(3) $a \times -b = -ab$ or $+a \times -b = -ab$.

(4) $-a \times b = -ab$ or $-a \times +b = -ab$.

These four examples merely express the rule for the concurrence of like and unlike signs. (Art. 569.)

(5) $5a \times 7b = 35ab$.

(6) $-7a \times -9b = 63ab$.

(7) $17x \times -19y = -323xy$.

(8) $\dfrac{-3x}{4} \times \dfrac{4y}{7} = \dfrac{-3xy}{7}$,

suppressing the number 4, which is common to the numerator and denominator. (Art. 76).

(9) $\dfrac{11abc}{12} \times \dfrac{-13a^2b^3c^4}{44} = \dfrac{-143a^3b^4c^5}{528} = \dfrac{-13a^3b^4c^5}{48}$,

suppressing the common factor 11.

(10) $11x \times -12y \times -13z = 1716xyz$.

(11) $\dfrac{2x}{3} \times \dfrac{-5xy}{8} \times \dfrac{9xyz}{10} = \dfrac{-3x^3y^2z}{8}$.

572. Case 2. When the multiplicand is a polynomial and the multiplier a mononomial.

Rule. "Multiply the single term of the multiplier into every successive term of the polynomial, and arrange the products of the several terms in the result, preceded by their proper signs, in any order which may appear most symmetrical or most convenient." Rule for Case 2.

"If the sign of the mononomial multiplier be negative, the signs of all the terms of the multiplicand will be changed: in every other respect the rule agrees with that which is given in Arithmetical Algebra (Art. 50).

Examples. 573. The following are examples:

(1) $a(a-b)=a^2-ab.$

(2) $-a^2(a^2-ab+b^2)=-a^4+a^3b-a^2b^2.$

(3) $xy(-x+y)=-x^2y+xy^2.$

(4) $\dfrac{3x}{4}\left(\dfrac{-x^2}{3}+\dfrac{2x}{5}-\dfrac{20}{9}\right)=\dfrac{-x^3}{4}+\dfrac{3x^2}{10}-\dfrac{5x}{3}.$

Rule for Case 3. 574. Case 3. When both the multiplicand and the multiplier are polynomials.

Rule. "Multiply successively every term of one factor into every term of the other, add the several partial products together, and arrange the terms of the result in any order which may be considered most convenient, without regard to the sign of the first term."

"If there be three factors, multiply the third into the product of the two first: and similarly for any number of them" (Art. 61).

Examples. 575. The following are examples:

(1)
$$\begin{array}{l} x-a \\ x-b \\ \hline x^2-ax \\ \quad\;\; -bx+ab \\ \hline x^2-(a+b)x+ab \end{array}$$

(2)
$$\begin{array}{l} \;\;3a-5b \\ -7a+4b \\ \hline -21a^2+35ab \\ \qquad\quad +12ab-20b^2 \\ \hline -21a^2+47ab-20b^2 \end{array}$$

(3) $(x+a)(x^2-ax+a^2)=x^3+a^3.$

(4) $(a^2-b^2+2bc-c^2)(-a^2+b^2+2bc+c^2)=-a^4+2a^2b^2+2a^2c^2$
$-b^4+2b^2c^2-c^4.$

(5) $(x^2+ax+b)(x^2-ax+c)=x^4-(a^2-b-c)x^2-(ab-ac)x$
$+bc.$

(6) $\left(\dfrac{5}{2}x^2+3ax-\dfrac{7a^2}{3}\right)\left(2x^2-ax-\dfrac{a^2}{2}\right)=5x^4+\dfrac{7ax^3}{2}-\dfrac{107a^2x^2}{12}$
$+\dfrac{5a^3x}{6}+\dfrac{7a^4}{6}.$

The examples of multiplication which are given in Arithmetical Algebra (Arts. 37, 51 and 69, and also Arts. 478 and 479) are examples likewise of Symbolical Multiplication, the processes followed being the same.

576. It is usual to arrange the terms of the product according to the powers of some one letter, which may be called the *letter* or *symbol of reference:* thus x is the *symbol of reference* in Examples 1, 3, 5, 6, and a in Example 2: the same letter would be the *symbol of reference* in Example 4, if the result was reduced to the equivalent form Letter or symbol of reference.

$$-a^4+2(b^2+c^2)a^2-b^4+2b^2c^2-c^4.$$

The same result may be equally arranged under the equivalent forms

$$-b^4+2(a^2+c^2)b^2-a^4+2a^2c^2-c^4$$

and

$$-c^4+2(a^2+b^2)c^2-a^4+2a^2b^2-b^4,$$

where b is the *symbol of reference* in one case, and c in the other.

577. The processes of division in Arithmetical and Symbolical Algebra merely differ in the additional rule which is requisite for determining the sign of the quotient or of its first term, when a negative sign affects one or both of the first terms (or the only terms when both of them are mononomials) of the dividend and divisor. Division in Symbolical Algebra.

There are three Cases of the operation of Symbolical Division corresponding to the three cases of the operation of multiplication (Art. 567): we shall consider them in their order. Three Cases of the operation of division.

578. Case 1. When both the dividend and divisor are mononomials. Rule for division when the divisor and dividend are mononomials.

Rule. "The sign of the quotient is positive or negative, according as the signs of the dividend and divisor are the same or different."

"The coefficient of the quotient is found by dividing the coefficient of the dividend by that of the divisor."

"The literal part of the quotient is found by obliterating the symbols or their powers, which are common to the dividend and divisor: and retaining those which are not thus suppressed." (Art. 78).

Examples. 579. The following are examples:

$$(1)\quad ax \div x = a: \text{ or, } \frac{ax}{x} = a.$$

$$(2)\quad -ax \div -x = a: \text{ or, } \frac{-ax}{-x} = a.$$

$$(3)\quad ax \div -x = -a: \text{ or, } \frac{ax}{-x} = -a.$$

$$(4)\quad -ax \div x = -a: \text{ or, } \frac{-ax}{x} = -a.$$

These four examples express the rule of signs: it is immediately derived from the rule for the *concurrence* of like and unlike signs in Multiplication (Art. 569), by observing that the product of the divisor and quotient is equal to the dividend (Arts. 70 and 72).

$$(5)\quad 9a^2x^3 \div 12a^3x^2: \text{ or, } \frac{9a^2x^3}{12a^3x^2}$$

$$= \frac{3 \times 3a^2x^2 \times x}{4 \times 3a^2x^2 \times a} = \frac{3x}{4a}:$$

suppressing the factor $3a^2x^2$, which is common to the dividend and divisor.

$$(6)\quad -18abcde \div -12bde, \text{ or}$$

$$\frac{-18abcde}{-12bde} = \frac{3ac}{2},$$

suppressing the factor $6bde$, which is common both to the dividend and divisor.

Other examples are given in Art. 77.

When the dividend is a polynomial and the divisor a monomial.

Case 2. When the dividend is a polynomial and the divisor a mononomial.

Rule. "Divide successively by the rule in Case 1 (Art. 578), every term of the dividend by the divisor, and the several results connected with their proper signs form the quotient." (Art. 81).

Examples. 580. The following are examples:

(1) Divide $-ax-bx$ by $-x$,

$$\frac{-ax-bx}{-x} = a+b.$$

(2) Divide $a x^4 - a^2 x^3 + a^3 x^2 - a^4 x$ by $-ax$,

$$\frac{a x^4 - a^2 x^3 + a^3 x^2 - a^4 x}{-a x} = -x^3 + a x^2 - a^2 x + a^3$$

$$= -(x^3 - a x^2 + a x - a^3) = a^3 - a^2 x + a x^2 - x^3,$$

these several forms being equivalent to each other.

(3) Divide $-12acfg + 4af^2g - 3fg^2h$ by $-4a^2b^2fg$,

$$\frac{-12acfg + 4af^2g - 3fg^2h}{-4a^2b^2fg} = \frac{3c}{ab^2} - \frac{f}{ab^2} + \frac{3gh}{4a^2b^2}.$$

Other examples are given in Art. 82.

581. Case 3. When the divisor is a polynomial.

The dividend may be either a polynomial or a mononomial: but in the latter case, as will be shewn hereafter, there can be no finite quotient. When the divisor is a polynomial.

Rule. "Arrange the divisor and dividend according to the powers of some one symbol or *letter of reference*, (Art. 576), or as much as possible in the same order of succession, whether alphabetical or otherwise, and place them in one line as in long division of numbers in arithmetic: find the quantity or expression which multiplied into the first term of the divisor will produce the first term of the dividend: this is the first term of the quotient: multiply this term into all the terms of the divisor and subtract the resulting product from the dividend: consider the remainder, if any, as a new dividend, and proceed as before, continuing the process until no remainder exists, or until the process becomes obviously interminable."

This is the same rule which is followed in Arithmetical Algebra, (Art. 84), and the examples which are given in illustration of it (Art 86), are examples also of Symbolical Division.

582. The following are additional examples: Examples.

(1) Divide $-2x^3 + 3x^2y - 6xy^2 + 9y^3$, by $x^2 + 3y^2$.

$$\begin{array}{l} x^2 + 3y^2)\ -2x^3 + 3x^2y - 6xy^2 + 9y^3\ (-2x + 3y \\ \qquad\qquad\ \ -2x^3 \qquad\quad\ \ - 6xy^2 \\ \hline \qquad\qquad\qquad\qquad\ + 3x^2y + 9y^3 \\ \qquad\qquad\qquad\qquad\ + 3x^2y + 9y^3 \\ \hline \end{array}$$

(2) Divide $x^4 - 2ax^3 + (a^2 - 9b^2)x^2 + 18ab^2x - 9a^2b^2$, by $x^2 - (a - 3b)x - 3ab$.

$$
\begin{array}{l|l}
x^2-(a-3b)x-3ab) & x^4-2ax^3+(a^2-9b^2)x^2+18ab^2x-9a^2b^2 \\
& \qquad\qquad\qquad\qquad (x^2-(a+3b)x+3ab \\
& \underline{x^4-(a-3b)x^3-3abx^2} \\
& \quad -(a+3b)x^3+(a^2+3ab-9b^2)x^2+18ab^2x \\
& \quad \underline{-(a+3b)x^3+(a^2-9b^2)x^2+3ab(a+3b)x} \\
& \qquad\qquad +3abx^2-(3a^2b-9ab^2)x-9a^2b^2 \\
& \qquad\qquad \underline{+3abx^2-(3a^2b-9ab^2)x-9a^2b^2}
\end{array}
$$

(3) Divide a by $1 - x$, (Art. 432).

$$
\begin{array}{l}
1-x)\ a\ (a+ax+ax^2+\&c. \\
\quad \underline{a-ax} \\
\qquad +ax \\
\qquad \underline{+ax-ax^2} \\
\qquad\qquad +ax^2 \\
\qquad\qquad \underline{+ax^2-ax^3} \\
\qquad\qquad\qquad +ax^3
\end{array}
$$

The dividend and the successive remainders are mononomials and the successive subtrahends are binomials: it follows therefore, that the remainders can never disappear, however far the operation is continued, and the series which forms the quotient is therefore interminable. (Art. 87).

(4) Divide $x + a$ by $x + b$.

$$
\begin{array}{l}
x+b)\ x+a\ \left(1+\dfrac{(a-b)}{x}-\dfrac{(a-b)b}{x^2}+\dfrac{(a-b)b^2}{x^3}-\&c.\right. \\
\quad \underline{x+b} \\
\qquad +(a-b) \\
\qquad \underline{(a-b)+\dfrac{(a-b)b}{x}} \\
\qquad\qquad -\dfrac{(a-b)b}{x} \\
\qquad\qquad \underline{-\dfrac{(a-b)b}{x}-\dfrac{(a-b)b^2}{x^2}} \\
\qquad\qquad\qquad +\dfrac{(a-b)b^2}{x^2}\ \ldots
\end{array}
$$

The terms of the divisor and dividend being arranged with reference to the letter x, we consider $a - b$, which is the first remainder, as forming a single term only.

If we reverse the order of the terms in the dividend and divisor, the process will stand as follows:

$$b + x)\ a + x\ \left(\frac{a}{b} - \frac{(a-b)\,x}{b^2} + \frac{(a-b)\,x^2}{b^3} - \text{\&c.}\right.$$

$$\begin{array}{l}
\qquad\qquad\qquad\qquad a + \dfrac{ax}{b} \\
\hline
x - \dfrac{ax}{b} = \left(1 - \dfrac{a}{b}\right)x = \dfrac{(b-a)\,x}{b} = -\dfrac{(a-b)\,x}{b} \\
\qquad\qquad\qquad\qquad -\dfrac{(a-b)\,x}{b} - \dfrac{(a-b)\,x^2}{b^2} \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{(a-b)\,x^2}{b^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{(a-b)\,x^2}{b^2} + \dfrac{(a-b)\,x^3}{b^3} \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \dfrac{(a-b)\,x^3}{b^3}.
\end{array}$$

We have indicated the successive reductions by which the first remainder is found: x and $\frac{ax}{b}$ are *like* terms (Art. 28), whose coefficients are 1 and $\frac{a}{b}$ respectively, (Art. 30): we then subtract the fraction $\frac{a}{b}$ from 1 or from its equivalent $\frac{b}{b}$, having the same denominator with $\frac{a}{b}$, (Art. 128), which gives us $\frac{b-a}{b}$ or its equivalent $-\frac{(a-b)}{b}$ for the coefficient of x in the remainder.

$$(5)\quad \frac{x-a}{x-b} = 1 - \frac{(a-b)}{x} - \frac{(a-b)\,b}{x^2} - \frac{(a-b)\,b^2}{x^3} - \text{\&c.}$$

All the terms of this series, after the first, are negative.

$$(6)\quad \frac{x-a}{x+b} = 1 - \frac{(a+b)}{x} + \frac{(a+b)\,b}{x^2} - \frac{(a+b)\,b^2}{x^3} + \text{\&c.}$$

$$(7)\quad \frac{a+x}{b-x} = \frac{a}{b} + \frac{(a+b)x}{b^2} + \frac{(a+b)x^2}{b^3} + \&c.$$

$$(8)\quad \frac{a-x}{b+x} = \frac{a}{b} - \frac{(a+b)x}{b^2} + \frac{(a+b)x^2}{b^3} - \&c.$$

$$(9)\quad \frac{x+5}{x+4} = 1 + \frac{1}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \&c.$$

$$(10)\quad \frac{10+x}{5+x} = 2 - \frac{x}{5} + \frac{x^2}{5^2} - \frac{x^3}{5^3} + \&c.$$

$$(11)\quad \frac{1-3x-2x^2}{1-4x} = 1 + x + 2x^2 + 2\times 4x^3 + 2\times 4^2x^4 + \&c.$$

$$(12)\quad \frac{x^2-px+q}{x-a} = x-(p-a) + \frac{k}{x} + \frac{ak}{x^2} + \frac{a^2k}{x^3} \ldots$$

where $k = a^2 - pa + q$.

In all these Examples it will be observed that the series proceeds according to *direct* or *inverse* powers of the *letter of reference,* according as its highest power occupies the last or the first place in the dividend and divisor.

Origin of interminable quotients.

583. Interminable quotients present themselves in Arithmetic, and also in Arithmetical and Symbolical Algebra, as necessary consequences of the Rule for division (Art. 88), which is equally applicable in all those sciences, whether the dividend is or is not divisible by the proposed divisor: in the first case, we get a terminable quotient, which multiplied into the divisor, reproduces the dividend: but in the second, there is no terminable quotient which multiplied into the divisor, will reproduce the dividend, however nearly it may approximate to it.

Interminable quotients in Arithmetic.

584. In Arithmetic, the arrangement of the digits of the divisor and the dividend, in the order of their magnitude, produces, in the case of interminable quotients, a series of imperfect quotients, which approximate more and more nearly to the true quotient the further the operation is continued: thus if we divide 1 by 3, we get the interminable or recurring quotient .333... and the successive imperfect quotients .3, .33, .333, .3333, will be found to differ, in defect, from the true quotient by

$$\frac{1}{3\times 10}{}^{*},\ \frac{1}{3\times 10^2},\ \frac{1}{3\times 10^3},\ \frac{1}{3\times 10^4}:$$

* For $3\left(.3 + \frac{1}{3\times 10}\right) = .9 + .1 = 1.$

and the difference for the nth quotient is $\frac{1}{3 \times 10^n}$: if therefore n be indefinitely great or if the operation be indefinitely continued, this difference will become indefinitely small or zero. (Art. 166).

585. In Arithmetical Algebra, we adopt or we assume a similar arrangement of the terms of the dividend and divisor in the order of their magnitude: thus in dividing $x^2 + a^2$ by $x - a$, we get the series

Interminable quotients in Arithmetical Algebra,

$$x + a + \frac{2a^2}{x} + \frac{2a^3}{x^2} + \ldots\ldots$$

whose successive terms, after the third, grow less and less continually, inasmuch as x is greater than a: but if we admit indifferently, as in Symbolical Algebra, all relations of the symbols and recognize therefore negative as well as positive expressions, we shall get, when x is less than a, by the same rule of operation, the *same* series in *form*

and in Symbolical Algebra.

$$x + a + \frac{2a^2}{x} + \frac{2a^3}{x^2} + \ldots\ldots$$

but whose successive terms go on increasing continually, instead of diminishing.

586. Series, such as are formed in Arithmetical Algebra, whose terms perpetually diminish and become ultimately zero, are called *convergent* series: whilst those whose terms perpetually increase, and which exclusively belong to Symbolical Algebra, are called *divergent* series: thus in the example considered in the last Article, the successive sums of two, three, four, or more terms of the *convergent* series will continually approximate more and more nearly to the value of the algebraical fraction in which it originates: but the corresponding aggregates of such terms, in the *divergent* series, will bear no such relation to it, being always different in sign and more and more different in magnitude, the further the operation is continued. In this case, therefore, (and the same remark will be found to be true generally), *the series becomes divergent, when the operation in which it originates ceases to belong to Arithmetical Algebra.*

Convergent and divergent series.

Their properties.

The transition from convergent to divergent series marks the transition from Arithmetical to Symbolical Algebra.

587. The operation of division in Arithmetical Algebra may introduce *negative* as well as *positive* terms in the quotient, by allowing the subtraction of partial products which are greater

Convergent and divergent series may origi-

nate in the same expression by a change in the arrangement of its terms.

as well as less than the successive remainders: thus if we divide x^2+a^2 by $x+a$, we get the series

$$x-a+\frac{2a^2}{x}-\frac{2a^3}{x^2}+\frac{2a^4}{x^3}-\&c.\ldots$$

which is *convergent* or *divergent* according as x is *greater* or *less* than a: but if we reverse the order of the symbols, dividing a^2+x^2 by $a+x$, we get the series

$$a-x+\frac{2x^2}{a}-\frac{2x^3}{a^2}+\frac{2x^4}{a^3}-\&c.\ldots$$

which is *convergent* or *divergent* according as a is *greater* or *less* than x: it follows, therefore, that though the fraction $\frac{x^2+a^2}{x+a}$ is identical in value with $\frac{a^2+x^2}{a+x}$, we must adopt, in Arithmetical Algebra, in dividing its numerator by the denominator, that form of the fraction which secures the arrangement of its terms in the order of their magnitude: the results of the operation, whenever this arrangement is departed from, belong exclusively to Symbolical Algebra.

Consequences of obliterating in certain cases, ∞ as well as zero.

588. If we multiply any number of the terms (n) of the quotient corresponding to $\frac{x^2+a^2}{x-a}$, by the divisor $x-a$, it will be found to differ in defect from the dividend x^2+a^2 by $\frac{2a^n}{x^{n-1}}$: and if we suppose the quotient series to be indefinitely continued or n to be indefinitely great, this difference will become zero (0) or infinity (∞), according as a is less or greater than x, or in other words, according as the series is a result both of Arithmetical and Symbolical Algebra, or of the latter science only: and if we were at liberty equally to neglect the symbols 0 and ∞ in the reverse process of passing from a series to the expression in which it originates, we might regard the series

$$x+a+\frac{2a^2}{x}+\frac{2a^3}{x^2}+\&c.$$

as equally significant for all relations of the values of the symbols: but under such circumstances, the processes which we must follow for the purpose of effecting this transition, must not be dependent upon the arithmetical aggregation of any number of the terms of the series, but upon methods which are altogether independent of the relative or absolute value of the symbols involved.

589. When symbols are incorporated by the operation of multiplication, the sign of the product is determined by the Rule of Signs (Art. 569): or in other words, the signs of the factors determine the sign of the product. When, therefore, the factors are assigned, the sign of the product is no longer arbitrary, and its relative interpretation, when affected by different signs, must be dependent upon the interpretation of the factors themselves: before entering, however, upon this inquiry, it will be necessary to determine the meaning of such products in Arithmetical Algebra, when their factors are not abstract numbers. Interpretation of algebraical products.

590. When one of the two symbols in the product ab is an abstract number, it simply means that one factor is to be multiplied by the other in its ordinary arithmetical sense, the units of the product being identical with those of the concrete factor: thus, if a denote a line of 6 inches in length, and b the number 7, then ab will denote a line of 6×7, or 42 inches in length: and inasmuch as the factors of the product ab are commutable with each other, without altering its value, we are at liberty to suppose the abstract factor to be in all cases the multiplier and the concrete factor the multiplicand*. When one of the two factors only is an abstract number.

591. When both the factors are concrete magnitudes, whether the same or different, their interpretation, when possible, must be made in conformity with the following principles. When both the symbols are concrete.

1st. *The product must be identical with the arithmetical product, when the factors are replaced by numbers.* Principles of interpretation.

For if not, the operations in Arithmetical Algebra would not be identical in meaning with the corresponding operations in arithmetic.

2nd. *The product is always the same in meaning and magnitude, when the factors incorporated are so.* For otherwise, the factors would not determine the product.

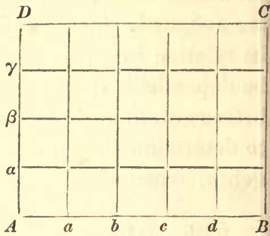
592. One of the most important cases which we have to consider, is that in which the symbols incorporated represent geometrical straight lines: it will be found that their product When the two symbols represent straight lines.

* Upon the same principle, if the concrete factor be affected with the sign $-$, the product will be affected with the same, and the *negative* units of one will be identical in meaning with the *negative* units of the other: thus if $-a$ denote a distance of 3 miles to the south of a given point, and b the number 5, their product $-ab$ or -15 will denote a distance of 15 miles to the south.

will correctly represent the area of the rectangle which those lines contain.

Their product is the rectangular area which they contain.

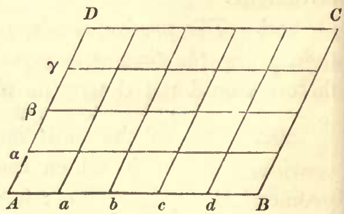
For, let us suppose the lines AB (a) and AD (b), which contain the rectangle $ABCD$, to be in the proportion of the numbers 5 and 4, and also to be represented by them: if we divide AB into 5 equal parts in the points a, b, c, d, and AD into 4 equal parts in the points α, β, γ, each of these equal parts will be a line represented by 1. Through the several points of division, let lines be drawn parallel to AB and to AD respectively, dividing the whole rectangular area into 5×4 or 20 equal squares, each of which is constructed upon one of the several linear units into which the sides are divided.



Assuming therefore the product ab to denote the rectangle $ABCD$, it will satisfy the first condition (Art. 591), because if we replace a and b by the numbers 5 and 4, then ab is replaced by their product or 20, the units in the factors being equal lines, and those in the product being the squares constructed upon them.

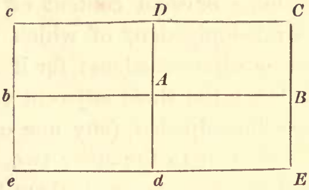
Again, all rectangles are equal which are contained by the same sides, and all products are equal which arise from the same factors: it follows, therefore, that in conformity with the second condition (Art. 591), the factors a and b determine the product ab, as well as the rectangle of which it is the symbolical expression*.

* A parallelogram, not rectangular, would satisfy the first condition, but not the second: for if the sides AB (5) and AD (4) of the oblique parallelogram $ABCD$ be divided into linear units, and lines parallel to the other sides be drawn through the points of division, it will be divided into a number of equal rhombs, which is equal to the numerical product of the numbers of linear units in the two adjacent sides: but inasmuch as parallelograms contained by the same sides, but making different angles with each other, have different areas, it follows that their products ab might correspond to different values when a and b were the same: or in other words, *the factors would not determine the product:* no such ambiguity exists with respect to the rectangle, where the angle included by the containing sides is determined by the definition of the figure.



593. Having shewn that, when a and b are geometrical straight lines, their product ab will express the area of the rectangle which they contain, it remains to consider its meaning, when one or both its factors become negative as well as positive. For this purpose, if we produce the adjacent sides BA and DA of the rectangle $ABCD$ to b and d respectively, making $Ab = AB$ and $Ad = AD$, and complete the rectangles $ADcb$, $ABEd$, $Abed$, we shall find, by the principles of interpretation established in the last chapter (Art. 558), that if $AB = a$, then $Ab = -a$, and if $AD = b$, then $Ad = -b$: and of the four adjacent rectangles which are thus formed, $ABCD$ is contained by $+a$ and $+b$, and expressed by $+ab$ or ab (Art. 592): $ADcb$ is contained by $-a$ and $+b$, and is therefore expressed by their product $-ab$: $ABEd$ is contained by $+a$ and $-b$, and is expressed by their product $-ab$: and lastly, $Abed$ is contained by $-a$ and $-b$, and is therefore expressed by $+ab$ or ab: for the expressions $+ab$ and $-ab$ represent equal magnitudes (Art. 558) with different signs, and those signs are determined by the rules of Symbolical Algebra: whilst the interpretation given to them corresponds equally to all values of the symbols involved, whether positive or negative.

Meaning of the product ab, when a and b are straight lines admitting of the signs + and −.

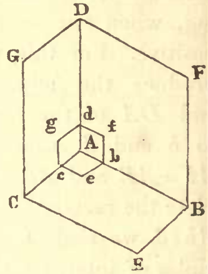


594. There are two rectangles $ABCD$ and $Abed$, which correspond to the same product ab; and two others $AbcD$ and $ABEd$, which equally correspond to the same product $-ab$: the two first arise from the product of $+a$ and $+b$ and of $-a$ and $-b$: the two last from the product of $-a$ and $+b$ and of $+a$ and $-b$. In passing therefore from the factors to the product, the product itself and its interpretation are fully determined; but in the inverse process, when we pass from the product to its component factors, there are two different pairs of factors which equally correspond to it, and there is no reason whatever for the selection of one pair of them in preference to the other: or in other words, the determination of the product is ambiguous. Thus the rectangles $ABEd$ and $ADcb$, which are both of them expressed by $-ab$, are equally related to the rectangles $ABCD$ and $Abed$, which are both of them expressed by $+ab$ or ab.

Ambiguity in passing from the product to the factors.

The product of three symbols representing lines, represents the solid parallelopipedon which they contain.

595. If we extend our inquiry to the determination of the meaning of the product abc of three factors $a, b, c,$ which severally represent geometrical straight lines, we shall find that it may correctly represent the volume or solid content of the rectangular parallelopipedon, of which a, b and c are three adjacent edges: for if AB (a), AC (b), AD (c), the three adjacent edges of such a parallelopipedon, (any one of which is perpendicular to the other two, or to the plane which passes through them,) are expressible by integral numbers, and are severally divided into equal linear units, (such as Ab, or Ac, or Ad), and if planes parallel to the bounding planes of the parallelopipedon were made to pass through the several points of division, the solid $ABFDGCE$ would be divided into cubes equal to each other (and to $becgdf$), constructed upon a linear unit, the number of which will be equal to the product of the numbers which express the number of linear units in each edge respectively: the requisite arithmetical condition is therefore satisfied, it being merely necessary to keep in mind that the units in the product are equal cubes, whilst those in the factors are equal lines.



This solid is of the same magnitude whatever be the order in which the symbols or the edges which they represent, are taken: and the product abc denotes a rectangular and not an oblique parallelopipedon, inasmuch as the latter solid involves the values of the angles which the edges make with each other, and which its definition does not determine.

Signs of the product of three factors, which may be positive or negative.

596. There are only two signs of the product of three factors, though they may arise from eight different combinations of the signs of the component factors; they are as follows:

(1) $+a \times +b \times +c = abc.$

(2) $+a \times -b \times -c = abc.$

(3) $-a \times +b \times -c = abc.$

(4) $-a \times -b \times +c = abc.$

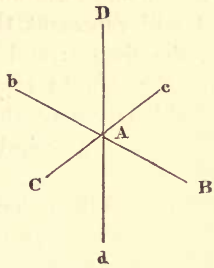
(5) $+a \times +b \times -c = -abc.$

(6) $+a \times -b \times +c = -abc.$

(7) $-a \times +b \times +c = -abc.$

(8) $-a \times -b \times -c = -abc.$

These eight products would correspond to eight different, though equal and similar rectangular parallelopipedons, having a common angle (A), and constructed upon edges which form severally one of each of the pairs of lines represented by $+a\,(AB)$ and $-a\,(Ab)$, $+b\,(AC)$ and $-b\,(Ac)$, $+c\,(AD)$ and $-c\,(Ad)$. Those pairs of solids, which touch by a common plane, and which have therefore two edges in common, have different signs: those pairs of solids which have one edge only in common, have the same sign: whilst those pairs which have only one point in common, and all whose three edges have therefore different signs, have also different signs. Their interpretation.



597. We should fail in attempting to give the interpretation of the meaning of the product of four or more symbols representing lines, or of two or more symbols representing areas, or of any other combinations of symbols representing lines, areas, or solids, which exceed three dimensions, inasmuch as there is no prototype in Geometry with which such products can be compared: in other words, the existence of such products is possible in symbols only. Products of four or more symbols.

598. We have assigned an interpretation to the products ab and abc, when the symbols, which they involve, represent geometrical lines, in conformity with the general principle which connects Symbolical with Arithmetical algebra, and which assumes that when the symbols are replaced by numbers, such products degenerate into ordinary arithmetical products: if we may suppose, therefore, lines to represent numbers, (and there is no relation of magnitude which they may not represent,) they may equally represent any concrete magnitudes whatsoever, of which these numbers are the representatives: it will follow, therefore, that if a, b and c are represented by lines, the rectangle contained by the lines a and b, and the rectangular parallelopipedon constructed upon the lines a, b and c, may represent any specific magnitudes, which ab and abc may represent, when a, b and c are replaced by numbers. We shall thus give to Geometry the character of a symbolical science. Geometry may be considered as a symbolical science.

Space described in uniform motion may be represented by a rectangular area.

599. Thus, if v represents the uniform velocity of a body's motion, and t the time during which it is continued, the product vt will represent the space over which the body has moved in the time t: and if we assume one line to represent v, and another line to represent t, the rectangle contained by them would represent the space described equally with the symbolical or numerical product vt.

Linear representations of units of different kinds perfectly arbitrary.

600. When, however, lines are assumed to represent quantities like velocity (v) and time (t), which are different in their nature, and therefore admit of no comparison with each other in respect of magnitude, the first assumption of them must be perfectly arbitrary: thus, if v denoting a certain velocity, be represented by an assumed line, v' denoting any other velocity, would be represented by another line bearing the same proportion to the former, that v' bears to v: and in a similar manner, one line may represent a time t, and another line any other time t', if they bear to each other the proportion of t to t': but the magnitudes v and t admit of no comparison with each other, and therefore the line which represents an assigned magnitude of one of them can bear no determinate relation to the line which represents an assigned magnitude of the other: in other words, the lines, which severally represent their units, may be assumed at pleasure.

Numerical units perfectly arbitrary.

601. The same remark applies, and for the same reasons, to the representation of essentially different quantities by means of numbers, the values of their primary units being perfectly arbitrary: thus the unit of time may be a second, a minute, an hour, &c. whilst the unit of space or velocity, (for one is the measure of the other) may be a foot, a yard, a mile, &c.: thus, if the units be assumed to denote severally a second of time and a foot in space, we may speak of a velocity denoted by 1, 2, 3, 10 or 20, being such as would cause a body to move uniformly over 1, 2, 3, 10 or 20 feet of space in one second, twice those spaces in 2 seconds, three times those spaces in 3 seconds, and therefore through a space which would be denoted by vt, if the body moved with a velocity equal to v, during any number of seconds denoted by t: and if we now pass from arithmetic to geometry, we may assume a line to represent a second of time, whilst an equal or any other line represents a foot in space or a velocity of one foot: such primary units being once

assumed, all other values of those magnitudes will be represented by lines bearing a proper relation to them.

602. The meaning of algebraical products, when the factors are any assigned quantities, being once determined, we experience no difficulty in interpreting the meaning of algebraical quotients, when the dividend and divisor are assigned both in representation and value: the general principle of such interpretations being, that "*the operation of Division is in all cases the inverse of that of Multiplication*:" in other words, the quotient or result of the division must be such a quantity, that, when multiplied into the divisor, it will produce the dividend: we will mention a few cases. Algebraical quotients.

603. If the dividend and divisor be both of them abstract numbers, the quotient is either an abstract number or a numerical fraction (Art. 92). Examples of their interpretation.

If the dividend be concrete and the divisor an abstract number or numerical fraction, the quotient is a concrete quantity of the same kind with the dividend (Art. 590).

If the dividend and divisor be concrete quantities of the same kind, the quotient is an abstract number or a numerical fraction (Art. 590).

If the dividend be an area and the divisor a line, the quotient is a line which contains, with the divisor, a rectangular area equal to the dividend, (Art. 592).

If the dividend be a solid and the divisor a line, the quotient is the rectangular base of an equal rectangular parallelopipedon, of which the divisor is the third edge (Art. 595).

If the dividend be a space described and the divisor the uniform velocity with which it is described, the quotient is the time of describing it (Art. 599).

It is not necessary, however, to multiply examples of such interpretations of the meaning of quotients, when the principle which connects them with their corresponding products admits of such easy and immediate application.

CHAPTER XIII.

ON THE DETERMINATION OF THE HIGHEST COMMON DIVISORS AND THE LOWEST COMMON MULTIPLES OF TWO OR MORE ALGEBRAICAL EXPRESSIONS.

Explanation of the meaning of highest common divisors and lowest common multiples of algebraical expressions.

604. THE determination of the *highest* common divisors and the *lowest* common multiples of two or more algebraical expressions will be required for the reduction of fractions to their most simple equivalent forms, in the same manner that the processes for finding the *greatest* common measure and the *least* common multiple of two or more numbers are involved in the corresponding reductions of numerical fractions (Arts. 98 and 116): we use the terms *highest* and *lowest*, with respect to the dimensions of the *symbol of reference* (Art. 576.) according to whose powers the terms of those expressions, whose common divisors or multiples are required, are arranged: the terms *greatest* and *least* ceasing to be applicable, in the case of expressions whose symbols are indeterminate in value.

Two algebraical expressions whose highest common divisor is required, may be assumed to form a fraction which is to be reduced to its most simple form.

605. We shall begin by considering the process for discovering the common divisor of two algebraical expressions only, and we shall arrange them as the numerator and denominator of a fraction, which it is proposed to reduce to its most simple form: for we have already shewn (Arts. 75 and 578.) that any factor common to the numerator and denominator of a fraction may be obliterated without altering its signification or value. There are several steps in this process, which it will be convenient to notice in their order.

Detection of simple numerical factors.

606. In the first place, there may exist a common numerical divisor of the coefficients of *all* the terms of the numerator and denominator, which may be discovered by inspection, or by the common arithmetical rule (Art. 98.) for that purpose: thus, 3 is a divisor of every term of the numerator and denominator of the fraction

Examples.

$$\frac{9x+15y}{12x^2+21y^2},$$

which is immediately reducible, therefore, to the more simple equivalent form

$$\frac{3x + 5y}{4x^2 + 7y^2}.$$

In a similar manner, the fraction

$$\frac{791\,a^2 + 452\,ab + 1017\,b^2}{1469\,a + 1243\,b}$$

is reducible to the more simple form

$$\frac{7a^2 + 4ab + 9b^2}{13a + 11b},$$

the common divisor being 113.

607. In the second place, there may exist a simple symbol, or powers or products of symbols, by which the several terms of the numerator and denominator may be divided *without introducing fractions*. Such divisors may generally be discovered by the mere inspection of the several terms in which they present themselves. Detection of mononomial algebraical factors.

Thus a is obviously a divisor of every term of the numerator and denominator of the fraction Examples.

$$\frac{a^2 - ax}{a^2 + ax},$$

which is therefore reducible to the more simple equivalent form

$$\frac{a - x}{a + x}.$$

Again, x^2y is a divisor of every term of the fraction

$$\frac{3x^4y - 4x^3y^2 + 5x^2y^3}{6x^3y + 7x^2y^2},$$

which becomes, when reduced,

$$\frac{3x^2 - 4xy + 5y^2}{6x + 7y}.$$

608. In the research of mononomial divisors, we confine ourselves to such as can be obliterated by division without introducing *fractional* terms: without such a restriction, there would be no limit to the number of such divisors, inasmuch as every algebraical expression can be divided by any mononomial, without producing an indefinite quotient. Thus, we may divide $a^2 - ax$ What class of mononomial divisors are considered as factors.

and $a^2 + ax$ severally by a^2, and their quotients $1 - \frac{x}{a}$ and $1 + \frac{x}{a}$ will form the fraction

$$\frac{1 - \frac{x}{a}}{1 + \frac{x}{a}},$$

which is equivalent to

$$\frac{a^2 - ax}{a^2 + ax} \text{ and } \frac{a - x}{a + x}:$$

but we should abstain from calling a^2 a common factor of the numerator and denominator of $\frac{a^2 - ax}{a^2 + ax}$, because, when applied as a divisor, it introduces the fractional term $\frac{x}{a}$, where no such fractional term previously existed.

Detection of compound algebraical factors.

609. Lastly, there may exist compound or polynomial expressions, which are divisors of the numerator and denominator of a fraction, by the discovery of which their dimensions may be reduced, and the form of the fraction generally simplified: thus, $x + a$ will be found to be a divisor of the numerator and denominator of the fraction

$$\frac{x^3 + a^3}{x^2 - a^2},$$

which is reducible therefore to the form

$$\frac{x^2 - ax + a^2}{x - a},$$

which is more simple than the former with respect to the dimensions, though not with respect to the number, of its terms: but in this, and in all similar cases, there is no obvious character presented to the eye, by which we can immediately discover such compound divisors when they exist, and we must follow, therefore, for this purpose, a process which is similar to that for finding the greatest common divisor of two numbers (Art. 98). It is as follows:

Rule.

"Arrange the numerator and denominator, or the expressions whose common divisor is required to be found, according to the powers of some one *symbol of reference* (Art. 576), and divide one of them by the other, making that

expression the divisor which does not involve the highest power of the symbol of reference: continue the division until the highest power of that symbol in the remainder is less than in the divisor: make this remainder the new divisor, and the last divisor the new dividend, and continue the same process until the remainder disappears, when the last divisor is the compound common divisor required: but if the first remainder from which the symbol of reference disappears is not identically equal to zero, then there exists no divisor in which that symbol is involved."

610. The application of this rule will very generally introduce coefficients of the symbol of reference, which are foreign to the most simple form of the compound divisor which is sought for, and which a subsequent modification (Art. 611) of the process will enable us to exclude: the following is an example of the occurrence of such a coefficient. Let it be required to reduce the fraction

Intrusion of extraneous multipliers of the compound factors determined by the Rule in Art. 609.

$$\frac{x^3+4x^2+5x+2}{x^2+5x+4}$$

to its most simple form.

$$\begin{array}{rl}
x^2+5x+4\,) & x^3+4x^2+5x+2\ (x-1 \\
& x^3+5x^2+4x \\ \hline
& \quad -x^2+\ \ x+2 \\
& \quad -x^2-5x-4 \\ \hline
\end{array}$$

$$\begin{array}{rl}
6x+6\,) & x^2+5x+4\ \left(\frac{x}{6}+\frac{2}{3}\right. \\
& x^2+\ \ x \\ \hline
& \quad +4x+4 \\
& \quad\ \ 4x+4 \\ \hline
& \quad\quad \cdot\ \ \cdot
\end{array}$$

The last divisor $6x+6$ divides the numerator and denominator of the fraction, giving finite or terminable quotients, but with *terms under a fractional form*: but if we had struck out the factor 6, which is common to both its terms, we should have obtained the form $x+1$ of the common divisor sought for, which

would reduce the original fraction to its most simple equivalent form, and which is

$$\frac{x^2 + 3x + 2}{x + 4}.$$

Modification of the Rule by which they may generally be excluded.

611. The intrusion of extraneous divisors in this and similar cases may generally be avoided by the following modification of the general Rule in Art. 609, the proof of which rests upon very simple principles.

"Multiply or divide the dividend and the successive divisors and dividends by any number, symbol or expression, which has no common divisor with the original dividend or divisor, so as to avoid, in every case, the introduction of fractional terms into the quotients, partial products or remainders: the last divisor found by the Rule in Art. 609, thus modified, will be generally the most simple common divisor in which the symbol of reference is involved.

Lemmas on which the preceding Rules are founded.

612. The preceding Rules (Arts. 609 and 611), are deducible from the following Lemmas.

LEMMA 1. If d divide A, it will divide Aa, provided a does not present itself under a fractional form.

For if $A = xd$, $Aa = axd$: and inasmuch as a does not present itself under a fractional form, there is no term in its denominator which can obliterate d or a factor of d.

LEMMA 2. If d divide A and B, it will divide $Aa \pm Bb$, provided that a and b do not present themselves under a fractional form.

For if $A = xd$ and $B = yd$, then $Aa = axd$ and $Bb = byd$: and therefore $Aa \pm Bb = axd \pm byd = (ax \pm by)\, d$: and there is no term in the denominators of a and b which can obliterate d or a factor of d.

LEMMA 3. The highest common divisor of A and B is the highest common divisor of Aa and Bb, if a and b have no common divisor, provided a and b do not present themselves under a fractional form.

For if $A = xd$ and $B = yd$, then x and y have no common divisor: in a similar manner, we have $Aa = axd$ and $Bb = byd$, when ax and by have no common divisor, and where neither a nor b can obliterate d or a factor of d.

LEMMA 4. The highest common divisor of Aa and Bb is the highest common divisor of A and B, if a and b have no common divisor, and do not present themselves under a fractional form.

For if $Aa = axd$ and $Bb = byd$, then $A = xd$ and $B = yd$: and since ax and by have no common divisor, it follows that d, which is the highest common divisor of Aa and Bb, is also the highest common divisor of A and B*.

It is assumed that a and B, and b and A have no common measure.

613. Two algebraical expressions, like two numbers, may be said to be *prime* to each other (Art. 107), if they have no common measure or divisor : a single algebraical expression may be said likewise to be absolutely *prime*, if it be not resolvible into rational factors. Extended use of the term *prime*.

614. The following is the general form of the process for finding the highest common divisor of two algebraical expressions; its proof will readily follow from the preceding Lemmas. Form and demonstration of the process.

$$
\begin{array}{llll}
B)\ A & & & \\
B)\ Aa\ (p & & & \\
\quad pB & & & \\
\hline
\quad Cc)\ B & & & \\
\qquad C)\ Bb\ (q & & & \\
\qquad\quad qC & & & \\
\hline
\qquad\quad Dd)\ C & & & \\
\qquad\qquad D)\ C\ (r & & & \\
\qquad\qquad\quad rD & & & \\
\hline
\qquad\qquad\quad \cdot\ \cdot & & &
\end{array}
$$

It is assumed that $a, b, c, d \ldots$ are prime to each other and to A and B (Art. 613), and that they do not present themselves under a fractional form.

We shall prove, in the first place, that every measure of A and B is a measure of D.

* It is assumed as a principle (which is proved in arithmetic and arithmetical algebra,) that a factor of a product can only be introduced through multiplication by that factor or by another which involves it: and that, when once introduced, it can only disappear through division : thus, if $a = cx$ and $b = dy$, then $ab = cdxy$, or in other words, c, d, x and y or all the separate factors of a and b are found in the product of a and b: and again, if $a = cx$ and $b = \frac{d}{x}$, then $ab = cd$, where the factor x, which existed in a, has disappeared through division, inasmuch as it presented itself also in the denominator of b.

For if any expression represented by x, measures A and B, it measures Aa and pB, and therefore $Aa-pB$ or Cc (Art. 612): and if x measures Cc, it also measures C, for c is prime to A and B, and therefore to x, which is a measure of A and B: and if x measures C and B, it measures Bb and qC, and therefore $Bb-qC$ or Dd: and if x measures Dd, it also measures D, for d is prime to A and B, and therefore to x, which is a measure of A and B.

In the second place, it may be shewn that D is a measure of A and B.

For D is a measure of qC and Dd, and therefore of $qC+Dd$ or Bb: and if D measures Bb, it measures B, for b is prime to A and B, and therefore to D: and if D measures pB and Cc, it measures $pB+Cc$ or Aa: and if D measures Aa, it measures A, for a is prime to A and therefore to D.

It follows, therefore, that every measure of A and B measures D, and that D measures A and B; and consequently D is the highest common divisor of A and B*.

When the expressions whose highest common divisor is required involve fractional terms.

615. If the expressions, whose highest common divisor is required, involve terms under fractional forms, it will generally be convenient to multiply all of them by a factor which is a common multiple (the lowest when discoverable) of their several denominators: the factor, thus introduced, whether simple or compound, may be obliterated, when convenient or necessary, at the conclusion of the process.

Other modifications of the preceding Rules, which are sometimes useful as tending to simplify the process, will be noticed amongst the following Examples.

Examples.

616. To reduce the fraction $\dfrac{x^3-a^3}{x^2-a^2}$ to its lowest terms.

$$
\begin{array}{lll}
 & x^2-a^2\big)\ x^3-a^3\ \big(x & \\
 & \quad\ \ x^3-a^2x & \\
\hline
\text{(Dividing by } a^2\text{)} & \quad\ \ a^2x-a^3 & \\
 & \qquad x-a\big)\ x^2-a^2\ \big(x+a & \\
 & \qquad\qquad\ x^2-ax & \\
\hline
 & \qquad\qquad\quad ax-a^2 & \\
 & \qquad\qquad\quad ax-a^2 & \\
\hline
 & \qquad\qquad\quad\ \cdot\ \ \cdot &
\end{array}
$$

* The process assumes that a, b, c, d... are so chosen, that the quotients p, q... may not take a fractional form: otherwise it would not necessarily follow that D would measure pB and qC, when it measures B and C. 612.)

The highest common divisor, therefore, of x^3-a^3 and x^2-a^2 is $x-a$: and the reduced fraction is

$$\frac{x^2+ax+a^2}{x+a}\,^*.$$

(2) To reduce the fraction $\dfrac{x^3-39x+70}{x^2-3x-70}$ to its lowest terms.

$$\begin{array}{llll}
x^2-3x-70\,\big) & x^3-39x+70 \;\big(x+3 & & \\
 & x^3-3x^2-70x & & \\
\hline
 & 3x^2+31x+70 & & \\
 & 3x^2-9x-210 & & \\
\hline
\text{(Dividing by 40)} & 40x+280 & & \\
 & x+7\,\big) & x^2-3x-70\;\big(x-10 & \\
 & & x^2+7x & \\
\hline
 & & -10x-70 & \\
 & & -10x-70 & \\
\hline
 & & \cdot\quad\cdot &
\end{array}$$

In this case $40x+280$ is the first remainder in which the highest power of x is less than the highest power of x in the divisor: its factor 40 is *prime* (Art. 613.) both to $x^2-3x-70$ and $x^3-39x+70$, and is therefore struck out: the reduced fraction is $\dfrac{x^2-7x+10}{x-10}$.

(3) To reduce the fraction

$$\frac{x^4-3x^3+x^2+3x-2}{4x^3-9x^2+2x+3}$$

to its lowest terms.

* Other examples are

(1) $\dfrac{x^4-a^4}{x^3+a^3}=\dfrac{x^3-ax^2+a^2x-a^3}{x^2-ax+a^2}$: the common divisor is $x+a$.

(2) $\dfrac{x^4+a^2x^2+a^4}{x^3-a^3}=\dfrac{x^2-ax+a^2}{x-a}$: the common divisor is x^2+ax+a^2

$$4x^3-9x^2+2x+3\,)\;x^4-3x^3+x^2+3x-2$$
$$4$$
$$4x^4-12x^3+4x^2+12x-8\;(x$$
$$4x^4-\;9x^3+2x^2+\;3x$$
$$-\;3x^3+2x^2+\;9x-\;8$$
$$-\;4$$
$$12x^3-\;8x^2-36x+32\;(3$$
$$12x^3-27x^2+\;6x+\;9$$
$$19x^2-42x+23\,)\;4x^3-9x^2+2x+3$$
$$19$$
$$76x^3-171x^2+38x+57\;(4x$$
$$76x^3-168x^2+92x$$
$$-\;3x^2-54x+57$$
$$-\;19$$
$$57x^2+1026x-1083\;(3$$
$$57x^2-\;126x+\;69$$

(Dividing by 1152)

$$1152x-1152$$
$$x-1\,)\;19x^2-42x+23\;(19x-23$$
$$19x^2-19x$$
$$-23x+23$$
$$-23x+23$$
$$\cdots\cdot$$

The reduced fraction is therefore

$$\frac{x^3-2x^2-x+2}{4x^2-5x-3}.$$

There are two divisions necessary in each of the two first stages of this operation, before we reach a remainder in which the highest power of (x), the symbol of reference, is less than in the divisor: in the first of these stages the multipliers 4 and -4*, and in the second, the multipliers 19 and -19 are required, in order to prevent the introduction of fractional quotients (Art. 611): these two pairs of multiplications might be severally replaced by one, if we had multiplied by 4^2 or 16 in one case, and by 19^2 or 361 in the other: the process would then stand as follows:

* We multiply by -4 instead of 4, in order to make the first term of the dividend positive; but this is not essential.

$$
\begin{array}{l}
4x^3-9x^2+2x+3)\ x^4-3x^3+x^2+3x-2 \\
\qquad\qquad\qquad\quad 16 \\
\hline
\qquad\qquad\qquad\quad 16x^4-48x^3+16x^2+48x-32\ (4x-3 \\
\qquad\qquad\qquad\quad 16x^4-36x^3+\ 8x^2+12x \\
\hline
\qquad\qquad\qquad -12x^3+\ 8x^2+36x\ -32 \\
\qquad\qquad\qquad -12x^3+27x^2-\ 6x\ -\ 9 \\
\hline
\qquad\qquad\qquad -19x^2+42x-23\ (4x^3-9x^2+2x+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad 361 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 1444x^3-3249x^2+\ 722x+1083\ (-76x+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad 1444x^3-3192x^2+1748x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad -57x^2-1026x\ +1083 \\
\qquad\qquad\qquad\qquad\qquad\qquad -57x^2-\ 126x\ +\ \ 69 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -1152x\ +1152
\end{array}
$$

It is obvious that the first form of the preceding process is more simple than the second, in consequence of its involving smaller coefficients*.

* The determination of a series of remainders in conformity with this rule, is made the foundation of Sturm's process for finding the number and limits of the real roots of numerical equations: the process, however, becomes extremely laborious, even for equations of the 4th or 5th degrees, in consequence of the rapidity with which the numerical coefficients increase.

Other examples of the same class are

(1) $$\frac{x^3-2x^2-15x+36}{3x^2-4x-15}=\frac{x^2+x-12}{3x+5}:$$

the common divisor is $x-3$.

(2) $$\frac{x^4-3x^3-18x^2+32x+96}{4x^3-9x^2-36x+32}=\frac{x^3+x^2-14x-24}{4x^2+7x-8}:$$

the common divisor is $x-4$.

(3) $$\frac{x^3-x^2-21x+45}{3x^2-2x-21}=\frac{x^2+2x-15}{3x+7}:$$

the common divisor is $x-3$.

(4) $$\frac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}=\frac{5x^2+1}{9x^3-4x}:$$

the common divisor is $3x+7$.

In this example, the denominator must be multiplied by 5, which will give 135 for the coefficient of the first term, which is the least common multiple of 27 and 15.

(5) $$\frac{x^4+2x^2+9}{7x^3-11x^2+15x+9}=\frac{x^2+2x+3}{7x+3}:$$

the common divisor is x^2-2x+3.

(6) $$\frac{12x^2+55x+63}{63x^3-36x^2-343x+196}=\frac{4x+9}{21x^2-61x+28}:$$

the common divisor is $3x+7$: the first multiplier is either 4 or 16.

(4) To reduce

$$\frac{x^3 - \dfrac{11a^2x}{18} - \dfrac{a^3}{9}}{x^2 - \dfrac{ax}{3} - \dfrac{2a^2}{3}}$$

to its lowest terms.

If the numerator and denominator be multiplied by 18 (Art. 615), the fraction becomes

$$\frac{18x^3 - 11a^2x - 2a^3}{18x^2 - 6ax - 12a^2}.$$

$$\begin{array}{rl}
18x^2 - 6ax - 12a^2 \,\big) & 18x^3 - 11a^2x - 2a^3 \quad (x \\
 & 18x^3 - 6ax^2 - 12a^2x \\
\hline
 & 6ax^2 + a^2x - 2a^3 \\
 & 3 \\
\hline
 & 18ax^2 + 3a^2x - 6a^3 \quad (a \\
 & 18ax^2 - 6a^2x - 12a^3 \\
\hline
\text{(Dividing by } 3a^2\text{)} & 9a^2x + 6a^3
\end{array}$$

$$\begin{array}{rl}
3x + 2a \,\big) & 18x^2 - 6ax - 12a^2 \; (6x - 6a \\
 & 18x^2 + 12ax \\
\hline
 & -18ax - 12a^2 \\
 & -18ax - 12a^2 \\
\hline
\end{array}$$

The fraction reduced is $\dfrac{6x^2 - 4ax - a^2}{6(x-a)}$: or, dividing its numerator and denominator by 6, it becomes

$$\frac{x^2 - \dfrac{2ax}{3} - \dfrac{a^2}{6}}{x - a}.$$

(5) To reduce the fraction

$$\frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc}.$$

to its lowest terms.

Let a be the symbol of reference:

$$\begin{array}{rl}
a^2-b^2-2bc-c^2) & a^2+2(b+c)a+b^2+2bc+c^2\ (1 \\
 & a^2 \qquad\qquad -b^2-2bc-c^2 \\ \hline
\{\text{Dividing by } 2(b+c)\} & 2(b+c)a+2(b^2+2bc+c^2) \\
a+b+c) & a^2-b^2-2bc-c^2\ \{a-(b+c) \\
 & a^2+(b+c)a \\ \hline
 & -(b+c)a-b^2-2bc-c^2 \\
 & -(b+c)a-b^2-2bc-c^2 \\ \hline
 & \cdot\quad\cdot\quad\cdot
\end{array}$$

The reduced fraction is $\frac{a+b+c}{a-b-c}$.

Let b be the symbol of reference:

$$\begin{array}{rl}
-b^2-2cb+a^2-c^2) & b^2+2(a+c)b+a^2+2ac+c^2\ (-1 \\
 & b^2+2cb-a^2+c^2 \\ \hline
(\text{Dividing by } 2a) & 2ab+2a^2+2ac \\
 & b+a+c, \text{ or } a+b+c,
\end{array}$$

which will be found to be the common divisor as before: the form of the process will be precisely similar if c be made the symbol of reference*.

(6) To reduce $\frac{ac+ad+bc+bd}{ae+af+be+bf}$ to its lowest terms.

Make a the symbol of reference:

$$(c+d)a+bc+bd\}\ (e+f)a+be+bf \qquad \{$$

* Other examples of the same class are

(1) $\frac{acx^2+(ad+bc)x+bd}{a^2x^2-b^2}=\frac{cx+d}{ax-b}$:

the common divisor is $ax+b$.

(2) $\frac{a^2x^2-2acxz-b^2y^2+c^2z^2}{a^2x^2+2abxy+b^2y^2-c^2z^2}=\frac{ax-by-cz}{ax+by+cz}$:

the common divisor is $ax+by-cz$.

(3) $\frac{a^3+(1+a)ax+x^2}{a^4-x^2}=\frac{a+x}{a^2-x}$:

the common divisor is a^2+x.

It will be found by trial, that $c+d$, or the coefficient of a divides the numerator but not the denominator: if we strike it out of the divisor, we get

$$\begin{array}{l} a+b\} \; (e+f)\,a+be+bf \;\{e+f \\ \qquad\quad\; \underline{(e+f)\,a+b(e+f)} \\ \qquad\qquad\qquad \cdot \qquad \cdot \end{array}$$

Therefore $\dfrac{ab+ad+bc+bd}{ae+af+be+bf}=\dfrac{c+d}{e+f}$.

(7) To reduce the fraction

$$\frac{4a^3cx-4a^3dx+24a^2bcx-24a^2bdx+36ab^2cx-36ab^2dx}{7abcx^3-7abdx^3+7ac^2x^3-7acdx^3-21b^2dx^3+21b^2cx^3+21bc^2x^3-21cbdx^3}$$

to its lowest terms.

In the first place, we discover by inspection, that $4ax$ is a factor of the numerator, and $7x^3$ a factor of the denominator: dividing the fraction by $\dfrac{4ax}{7x^3}$ or $\dfrac{4a}{7x^2}$, in order to simplify the remaining part of the operation, and arranging the result according to powers of a, we get

$$\frac{(c-d)\,a^2+6\,(bc-bd)\,a+9\,(b^2c-b^2d)}{(bc-bd+c^2-cd)a-3(b^2d-b^2c-bc^2+cbd)}.$$

In the next place, we find by trial that $c-d$, the coefficient of a^2, is a common divisor of the numerator and denominator: the fraction reduced, by dividing by it, becomes

$$\frac{a^2+6ab+9b^2}{(b+c)\,a+3\,(b^2+bc)}.$$

Arranging the process in the usual form, we get

$$(b+c)\,a+3\,(b^2+bc)\,)\; a^2+16ab+9b^2\;($$

It will be found that $b+c$, the coefficient of a in the divisor, is a factor of the divisor but not of the dividend: dividing the divisor by it, we get

$$\begin{array}{l} a+3b)\; a^2+6ab+9b^2\;(a+3b \\ \qquad\quad\;\; \underline{a^2+3ab\qquad\qquad} \\ \qquad\qquad\quad\; 3ab+9b^2 \\ \qquad\qquad\quad\; \underline{3ab+9b^2} \\ \qquad\qquad\qquad \cdot \quad \cdot \end{array}$$

It follows, therefore, that

$$\frac{a^2+6ab+9b^2}{(b+c)\,a+3\,(b^2+bc)}=\frac{a+3b}{b+c},$$

and the original fraction, in its lowest terms, replacing the factor $\frac{4a}{7x^2}$, is

$$\frac{4a\,(a+3b)}{7x^2\,(b+c)},$$

the highest common divisor being

$$x\,(c-d)\,(a+3b), \text{ or } acx-adx+3bcx-3bdx\,^*.$$

617. The preceding process will always succeed in detecting the common factors of two algebraical expressions, when exhibited under a rational form, whenever they exist.

The common factors of two algebraical expressions under a rational form may always be found.

For in the first place, when the same power only of each symbol presents itself (Ex. 6, Art. 616), the common factor, if any, will be found amongst the successively reduced coefficients of such symbols.

In the second place, when different powers of the symbol of reference present themselves, the common factors, if any, which are independent of that symbol, are factors of its coefficients, and may be found by the same general rule†.

* The following are other examples of the same class:

(1) $$\frac{6xy+8x+9y+12}{10xy-8x+15y-12}=\frac{3y+4}{5y-4}.$$

(2) $$\frac{xyz+3xy+2xz+yz+6x+3y+2z+6}{xyz+2xy+2xz+2yz+4x+4y+4z+8}=\frac{xz+3x+z+3}{xz+2x+2z+4}.$$

The *primary* coefficients of x in the numerator and denominator are

$$yz+3y+2z+6 \text{ and } yz+2y+2z+4:$$

the *secondary* coefficients of y in the *primary* coefficients of x are $z+3$ and $z+2$, which are not identical: but the *secondary* coefficients of z derived from the same *primary* coefficients of x are $y+2$ and $y+2$, which are identical, and form the common divisor required. The coefficients $yz+3y+2z+6$, $z+3$, $y+2$ are the successively *reduced* coefficients of x in the numerator, amongst which the common divisor, if any, which is independent of x, must be found.

† Thus the terms of the fraction

$$\frac{(a^2-b^2)\,x^2-(2a^2-ab-b^2)\,bx+a\,(a-b)\,b^2}{(a-b)^2x^2-(2a^2-3ab+b^2)\,x+a^2b^2-ab^3}$$

are arranged according to the primary symbol of reference x: we first find $a-b$, which is the common measure of the coefficients a^2-b^2 and $(a-b)^2$ of the first terms of the numerator and denominator, and which is found to be a common measure of all their other terms: we subsequently find, by the general rule, $x-b$

When these factors, if any, are struck out, we finally determine, by the same rule, the common factors which involve the symbol of reference, and thus obtain the most simple equivalent form which the primitive fraction admits.

Rule for finding the highest common divisor of three or more algebraical expressions.

618. If the highest common factor of three algebraical expressions A, B and C is required, we find X the highest common divisor of A and B; and then the highest common divisor of X and C is the highest common divisor of A, B and C: the proof is the same, *mutatis mutandis*, as of the Rule for finding the greatest common measure of three numbers (Art. 106); and similarly for any number of such expressions.

Extended use of the term multiple.

The lowest common multiple.

619. An algebraical expression, which is divisible by another, may be said to be a *multiple* of it, and the *lowest* or most simple *common multiple* of two or more such expressions is required in the reduction of two or more algebraical fractions to their most simple equivalent forms, in the same manner as the least common multiple of two or more numbers is required in the corresponding reductions of numerical fractions: for this purpose, we divide their product by their highest common divisor* (Art. 116): and when there are three or more of such ex-

to be a common measure of the numerator and denominator of the reduced fraction

$$\frac{(a+b)x^2-(2a+b)bx+ab^2}{(a-b)x^2-(2a-b)bx+ab^2},$$

leading to the most simple form of the equivalent fraction, which is

$$\frac{(a+b)x-ab}{(a-b)x-ab}.$$

* If x be the highest common divisor of A and B, and if $A=ax$ and $B=bx$, where a and b are prime (Art. 613) to each other, then $\frac{AB}{x}=Ab=Ba$, is the lowest common multiple (M) of A and B: for if not, let m be a multiple of A and B, whose dimensions are lower than those of M, and let $M=my=Ab=Ba$: it follows, therefore, that $m=A\times\frac{b}{y}=B\times\frac{a}{y}$; or in other words, that y is a common divisor of a and b, which is contrary to the hypothesis.

It is the fundamental principle of the theory of the measures and multiples of numbers that divisors can only be introduced by multiplication, and obliterated by division: in other words, that ab can have no divisor, which is not a product of divisors of a and b, where 1 and a, 1 and b are included amongst them: and this principle is so essentially involved in our primary conception of number, that little can be gained by any attempt to establish it by a formal demonstration, as in Art. 110.

pressions, whose *lowest common multiple* is required, we apply the same rules, *mutatis mutandis*, as in common arithmetic, (Arts. 118, 120, 121.)

620. The following are examples:

(1) The lowest or the most simple common multiple of $x^2 - a^2$ and $x^3 - a^3$ is Examples.

$$\frac{(x^2 - a^2)(x^3 - a^3)}{x - a} = (x + a)(x^3 - a^3) = x^4 + ax^3 - a^3x - a^4.$$

(2) The lowest common multiple of $x^3 - 3x^2 + 7x - 21$ and $x^4 - 49$ is

$$\frac{(x^4 - 49)(x^3 - 3x^2 + 7x - 21)}{x^2 + 7} = (x^4 - 49)(x - 3)$$

$$= x^5 - 3x^4 - 49x + 147.$$

(3) The lowest common multiple of

$$x^2 - a^2, \; x^2 - (a + b)x + ab \text{ and } x^2 - b^2$$

is $x^4 - (a^2 + b^2)x^2 + a^2b^2$.

This may be found, *mutatis mutandis*, by the rule in Art. 126, as follows:

$$\begin{array}{r|lll} x - a & x^2 - a^2, & x^2 - (a + b)x + ab, & x^2 - b^2 \\ \hline x - b & x + a, & x - b & , x^2 - b^2 \\ \hline & x + a, & 1 & , x + b. \end{array}$$

The product of $x - a$, $x - b$, $x + a$, $x + b$ is the lowest common multiple required.

CHAPTER XIV.

ON THE REDUCTION OF ALGEBRAICAL EXPRESSIONS TO THEIR MOST SIMPLE EQUIVALENT FORMS.

Reduction of single fractions.

621. WHEN an algebraical expression presents itself under a fractional form, whose numerator and denominator admit of a common divisor, their dimensions may be lowered and the form of the fraction generally simplified without altering its signification or value, by the process which is given in the last Chapter.

Reduction of two or more fractions, connected by the signs + and − to their most simple equivalent form.

622. When two or more algebraical expressions, one or more of which are under a fractional form, are connected by the signs + and −, they may be reduced to their lowest common denominator, and subsequently added or subtracted or reduced to their most simple equivalent form, by the same rule, *mutatis mutandis,* which is given for the addition and subtraction of numerical fractions in Art. 124: it is as follows.

Rule.

RULE. "*Find the* LOWEST *common multiple* (Art. 619) *of the denominators of all the fractions, for the new common denominator.*

Find the successive quotients which arise from dividing this LOWEST *common multiple by the several denominators of the fractions and multiply them successively into the numerators of the several fractions, thus forming the successive numerators of the equivalent fractions with a common denominator; connect the new numerators together with their proper signs, and beneath the result write the common denominator, reducing the fraction, which they form, to its lowest terms.*" *

Examples.

623. The following are examples.

(1) To reduce the expression $\frac{a+b}{a-b} + \frac{a-b}{a+b}$ to its most simple equivalent form.

The lowest common multiple of the denominators is their product $a^2 - b^2$:

$$\frac{(a^2-b^2)}{a-b} \times (a+b) = (a+b)(a+b) = a^2 + 2ab + b^2,$$

$$\frac{(a^2-b^2)}{a+b} \times (a-b) = (a-b)(a-b) = a^2 - 2ab + b^2$$

$$\text{Their sum} \qquad = 2a^2 + 2b^2$$

* See Appendix.

The final equivalent fraction is therefore

$$\frac{2a^2+2b^2}{a^2-b^2} \text{ or } \frac{2(a^2+b^2)}{a^2-b^2},$$

which admits of no further reduction.

(2) To reduce the expression $\dfrac{x+6}{x^2+4x-21}-\dfrac{x+2}{x^2-9}$ to it most simple equivalent form.

The lowest common multiple of the denominators (D) is

$$x^3+7x^2-9x-63.$$

$$\frac{D(x+6)}{x^2+4x-21}=(x+3)(x+6) \;=x^2+9x+18,$$

$$\frac{-D(x+2)}{x^2-9} \;=-(x+7)(x+2)=-x^2-9x-14$$

Their sum $= \quad 4$

The final equivalent fraction is therefore

$$\frac{4}{x^3+7x^2-9x-63},$$

which admits of no further reduction.

(3) To reduce the expression $\dfrac{x}{1-x}-\dfrac{x^2}{1-x^2}-\dfrac{x^3}{1-x^3}$ to its most simple equivalent form.

$$D=(1+x)(1-x^3)=1+x-x^3-x^4.$$

$$\frac{D\times x}{1-x}=x(1+2x+2x^2+x^3)=x+2x^2+2x^3+x^4,$$

$$\frac{D\times -x^2}{1-x^2}=-x^2(1+x+x^2) \quad = \quad -x^2-x^3-x^4,$$

$$\frac{D\times -x^3}{1-x^3}=-x^3(1+x) \quad = \quad -x^3-x^4,$$

Their sum $= x+x^2-x^4$

The final equivalent fraction is therefore

$$\frac{x+x^2-x^4}{1+x-x^3-x^4}=\frac{x(1+x-x^3)}{1+x-x^3-x^4},$$

which admits of no further reduction.

(4) To reduce $\dfrac{1}{(x+1)^3}-\dfrac{3}{2(x+1)^2}+\dfrac{5}{4(x+1)}-\dfrac{5}{4(x+3)}$ to its most simple equivalent form.

$$D = 4(x+1)^3(x+3).$$

$$\frac{D \times 1}{(x+1)^3} = 4 \times (x+3) = 4x + 12,$$

$$\frac{D \times -3}{2(x+1)^2} = -6(x+1)(x+3) = -6x^2 - 24x - 18,$$

$$\frac{D \times 5}{4(x+1)} = 5(x+1)^2(x+3) = 5x^3 + 25x^2 + 35x + 15,$$

$$\frac{D \times -5}{4(x+3)} = -5(x+1)^3 = -5x^3 - 15x^2 - 15x - 5$$

$$\overline{\qquad 4x^2 \qquad + 4 \qquad}$$

The final equivalent fraction is therefore

$$\frac{4x^2 + 4}{4(x+1)^3(x+3)} = \frac{x^2+1}{(x+1)^3(x+3)},$$

which admits of no further reduction.

In the preceding reductions we have adhered strictly to the general rule, though the same result can frequently be obtained by shorter and more expeditious processes, which it is not necessary to notice: they can only be safely employed by a student who has already become familiar with algebraical operations, and whose memory is stored with an habitual knowledge of a great number of their more simple and elementary results*.

* The following are examples of the same class:

(1) $\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{4ab}{a^2-b^2}$.

(2) $\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$.

(3) $\frac{1}{x-6} - \frac{1}{x-5} = \frac{1}{x^2-11x+30}$.

(4) $\frac{1}{1-x} - \frac{2}{1-x^2} = -\frac{1}{1+x}$.

(5) $\frac{a}{c} - \frac{(ad-bc)x}{c(c+dx)} = \frac{a+bx}{c+dx}$.

(6) $\frac{b}{d} + \frac{ad-bc}{d(c+dx)} = \frac{a+bx}{c+dx}$.

(7) $\frac{1}{3(1+x)} - \frac{x-2}{3(1-x+x^2)} = \frac{1}{1+x^3}$.

(8) $\frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{2(1+x^2)} = \frac{1}{1-x^4}$.

Multiplication and division of fractions.

624. The rule for the multiplication and division of fractions is given in Art. 143: and it will be seen, by a reference to the Articles which precede it, that it is derived by a species of anticipation, from the principles which are made the foundation of Symbolical Algebra: those principles, as applicable to the cases under consideration, may be restated as follows.

The product of two fractions.

625. What is the product of $\frac{a}{b}$ and $\frac{c}{d}$?

Supposing $\frac{c}{d}$ to be the multiplier, we may assume that amongst the successive values of c there is one which is equal to md, where m is a whole number, making $\frac{c}{d}$ equal to $\frac{md}{d}$ or m: under such circumstances, therefore, the product of $\frac{a}{b}$ and $\frac{c}{d}$ becomes the product of $\frac{a}{b}$ and m, which is $\frac{am}{b}$ (Art. 130): such would be the result if the general form of the product of $\frac{a}{b}$ and $\frac{c}{d}$ was $\frac{ac}{bd}$, and no other form of this product will satisfy the required condition: and inasmuch as it is assumed that the form of this product, whatever it may be, is independent of the specific values of the symbols involved (Art. 546), it follows that $\frac{ab}{cd}$, which is the form of the product of $\frac{a}{b}$ and $\frac{c}{d}$ in one case, must be its form likewise in all others.

Quotient of one fraction divided by another.

626. What is the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$?

Supposing the divisor $\frac{c}{d}$, as before, to become $\frac{md}{d}$ or m, where m is a whole number, it will follow that the quotient of $\frac{a}{b}$ by $\frac{c}{d}$ will become, under such circumstances, identical with

(9) $$\frac{1}{8(x-1)} - \frac{1}{4(x-3)} + \frac{1}{8(x-5)} = \frac{1}{x^3 - 9x^2 + 23x - 15}.$$

(10) $$\frac{1}{(a-b)(x+b)} - \frac{1}{(a-b)(x+a)} - \frac{1}{(x+a)^2} = \frac{a-b}{(x+a)^2(x+b)}.$$

(11) $$\frac{a^2}{(b-a)(c-a)(x+a)} + \frac{b^2}{(a-b)(c-b)(x+b)} + \frac{c^2}{(a-c)(b-c)(x+c)} = \frac{x^2}{(x+a)(x+b)(x+c)}.$$

the quotient of $\frac{a}{b}$ by m, which is $\frac{a}{bm}$ (Art. 131): such would be the result, if the form of the quotient of $\frac{a}{b}$ by $\frac{c}{d}$ was $\frac{ad}{bc}$, and no other form of this quotient will satisfy the required condition: and inasmuch as it is assumed that the form of this quotient, whatever it may be, is independent of the specific values of the symbols involved, it follows that $\frac{ad}{bc}$, which is the form of the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$ in one case, must be its form likewise in all others.

Examples. 627. The following are examples of the multiplication and division of fractions.

(1) Multiply $\frac{x^2-9}{x+4}$ and $\frac{x-1}{x+3}$.

$$\frac{x^2-9}{x+4} \times \frac{x-1}{x+3} = \frac{(x^2-9)(x-1)}{(x+4)(x+3)} = \frac{x^3-x^2-9x+9}{x^2+7x+12} = \frac{x^2-4x+3}{x+4},$$

when reduced to its lowest terms.

(2) Multiply $\frac{a^2+b^2}{a^2-b^2}$ and $\frac{a-b}{a+b}$.

$$\frac{a^2+b^2}{a^2-b^2} \times \frac{a-b}{a+b} = \frac{a^3-a^2b+ab^2-b^3}{a^3+a^2b-ab^2-b^3} = \frac{a^2+b^2}{a^2+2ab+b^2},$$

when reduced to its lowest terms.

(3) Divide $\frac{x^2-4}{x^2+2}$ by $\frac{x+2}{x+1}$.

$$\frac{x^2-4}{x^2+2} \div \frac{x+2}{x+1} = \frac{(x^2-4)(x+1)}{(x^2+2)(x+2)} = \frac{x^3+x^2-4x-4}{x^3+2x^2+2x+4} = \frac{x^2-x-2}{x^2+2},$$

when reduced to its lowest terms.

(4) Divide $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$.

$$\frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b} = \frac{(a^2+b^2)(a+b)}{(a^2-b^2)(a-b)} = \frac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3} = \frac{a^2+b^2}{a^2-2ab+b^2},$$

when reduced to its lowest terms.

Division by $\frac{1}{a}$ equivalent to multiplication by a.

628. The quotient of the division of a by $\frac{1}{b}$ is ab, and the product of a and $\frac{1}{b}$ is $\frac{a}{b}$ (Art. 144.): it follows, therefore,

that we change the operation of division into multiplication by *inverting* the divisor, and the operation of multiplication into division by *inverting* the multiplier.

629. By thus changing the character of the operations performed by inverting the symbols or expressions involved in them, or, in other words, by transferring them from the numerator to the denominator of a fraction or conversely, we shall not only be enabled to produce an almost endless variety of equivalent forms, but likewise, in many cases, to reduce them, when given, to others which are more simple and convenient: the following are examples of such conversions: Great varieties of equivalent forms, with examples of their formation and reduction.

$$(1)\quad \frac{1}{\left(\frac{1}{a}\right)} = 1 \div \frac{1}{a} = a.$$

$$(2)\quad \frac{\left(\frac{1}{1}\right)}{a} = \frac{1}{a}:\quad \text{for } \left(\frac{1}{1}\right) = 1.$$

$$(3)\quad \frac{\left(\frac{1}{1}\right)}{\left(\frac{1}{a}\right)} = \frac{1}{1} \div \frac{1}{a} = 1 \div \frac{1}{a} = a.$$

$$(4)\quad \frac{\frac{1}{1}}{\left(\frac{1}{a}\right)} = 1 \div \frac{1}{\left(\frac{1}{a}\right)} = 1 \div a = \frac{1}{a}.$$

The student should observe very carefully the import of the brackets in these four examples (Art. 24), as they will be found sufficient to guide him in the interpretation of their meaning and use in similar cases.

$$(5)\quad \frac{\dfrac{1}{1+x} + \dfrac{x}{1-x}}{\dfrac{1}{1-x} - \dfrac{x}{1+x}} = 1:$$

for the numerator $\dfrac{1}{1+x} + \dfrac{x}{1-x} = \dfrac{1+x^2}{1-x^2}$, and the denominator $\dfrac{1}{1-x} - \dfrac{x}{1+x} = \dfrac{1+x^2}{1-x^2}$: their quotient is therefore 1.

$$(6) \quad \frac{\dfrac{1}{1+x} + \dfrac{1}{1-x}}{\dfrac{1}{1-x} - \dfrac{1}{1+x}} = \frac{1}{x}:$$

following the same process of reduction as in Ex. 5.

$$(7) \quad \frac{\dfrac{1}{1+x}}{1 - \dfrac{1}{1+x}} = \frac{1}{x}: \text{ for } 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x},$$

and the fraction becomes, therefore $\dfrac{\left(\dfrac{1}{1+x}\right)}{\left(\dfrac{x}{1+x}\right)} = \dfrac{1}{x}$.

$$(8) \quad \frac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}} = \frac{1+x}{1+2x}: \text{ for } \frac{1}{1+\cfrac{1}{x}} = \frac{x}{1+x};$$

and, therefore, $\dfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}} = \dfrac{1}{1+\dfrac{x}{1+x}} = \dfrac{1+x}{1+2x}$,

by multiplying the numerator and denominator by $1+x$.

$$(9) \quad \frac{1}{\dfrac{1}{x-4} - \dfrac{1}{x-3}} = x^2 - 7x + 12: \text{ for}$$

$$\frac{1}{x-4} - \frac{1}{x-3} = \frac{x-3-x+4}{x^2-7x+12} = \frac{1}{x^2-7x+12}.$$

$$(10) \quad \frac{1}{\dfrac{1}{x+3} + \dfrac{1}{x+5} + \dfrac{1}{x-8}} = \frac{x^3-49x-120}{3x^2-49}.$$

$$(11) \quad \frac{1}{1+\cfrac{1}{x+\cfrac{1}{x}}} = \frac{1+x^2}{1+x+x^2}:$$

for multiplying the numerator and denominator by $x + \dfrac{1}{x}$, we get

$$\frac{x+\dfrac{1}{x}}{x+\dfrac{1}{x}+1} = \frac{1+x^2}{1+x+x^2}.$$

$$(12) \quad \cfrac{x}{1+\cfrac{x}{1+x+\cfrac{x}{1+x+x}}} = \frac{x+3x^2+2x^3+x^4}{1+4x+3x^2+2x^3}.$$

CHAPTER XV.

FORMAL STATEMENT OF THE PRINCIPLE OF THE PERMANENCE OF EQUIVALENT FORMS.

630. In the exposition of the fundamental operations of addition, subtraction, multiplication and division in Symbolical Algebra, we have adopted the corresponding rules of operation in Arithmetical Algebra, extending them to all values of the symbols involved, as well as to those additional derivative forms* which are the necessary results of that extension: and we have subsequently endeavoured to give to those extended operations and to their results, such an interpretation as was consistent with the conditions which they were required to satisfy. In the further developement of this science we shall continue to be guided by the same principle, making the results of defined operations, or the rules for forming them, the basis of the corresponding operations and results in Symbolical Algebra, and also of the interpretation of the meaning which must be given to them, whenever such interpretation is practicable. The general principle followed in the discovery or determination of equivalent forms.

631. This principle, which is thus made the foundation of the operations and results of Symbolical Algebra, has been called "The principle of the permanence of equivalent forms†", and may be stated as follows: Its formal statement

"*Whatever algebraical forms are equivalent, when the symbols are general in form but specific in value, will be equivalent likewise when the symbols are general in value as well as in form.*"

It will follow from this principle, that all the results of Arithmetical Algebra will be results likewise of Symbolical Algebra: and the discovery of equivalent forms in the former science, possessing the requisite conditions, will be not only their discovery in the latter, but the *only* authority for their existence: for there are no definitions of the operations in Symbolical Algebra, by which such equivalent forms can be determined‡. and consequences.

* Such are $+a$ and $-a$, and other forms which are thus derived, and which are not recognized in Arithmetical Algebra.

† Treatise on Algebra, p. 105. Cambridge, 1830.

‡ See Appendix.

Meaning of the term operation.

632. The term *operation* is used in the most comprehensive sense, as including every process by which we pass from one equivalent form to another: and its interpretation, in Symbolical Algebra, as we have already seen, will more or less change with every change of the circumstances of its application: the interpretations which we have given of the operations of addition and subtraction, multiplication and division, in the preceding Chapters, furnish examples of such changes.

CHAPTER XVI.

THE THEORY OF INDICES.

633. The continued product of a number or symbol a into itself, repeated as a factor n times, is expressed by a^n (Arts. 38 and 39) and the expression itself is called a *power* (the n^{th}) of a, and n its *exponent* or *index:* and it is easily shewn to be a necessary consequence of this definition, that the product of two *powers* of the same symbol is also a *power* of that symbol, whose *index* is equal to the sum of the *indices* of the component factors: that is, if a^m and a^n be two *powers* of a, then (Art. 44) Definitions of indices and powers in Arithmetical Algebra, and consequences to which they lead.

$$a^m \times a^n = a^{m+n}.$$

It will follow from this conclusion and the known relations of the operations of multiplication and division, that, if m be greater than n, $\frac{a^m}{a^n} = a^{m-n}$: for the product of the divisor a^n and the quotient a^{m-n}, or $a^n \times a^{m-n} = a^{n+m-n} = a^m$; we conclude therefore that a^{m-n} is the quotient required, inasmuch as, when multiplied into the divisor, it produces the dividend.

634. The definition of a *power,* in Arithmetical Algebra, implies that its index is a whole number: and if this condition be not fulfilled, the definition has no meaning, and therefore no conclusions are deducible from it: the principles, however, of Symbolical Algebra, will enable us, not merely to recognise the existence of such powers, but likewise to give, in many instances, a consistent interpretation of their meaning. The index of a power in Arithmetical Algebra is necessarily a whole number.

635. Observing that the indices m and n in the expressions which constitute the equation The equation in Art. 633 generalized by the principle of the permanence of equivalent forms.

$$a^m \times a^n = a^{m+n},$$

though *specific* in value, are *general* in form, we are authorized to conclude by "the principle of the permanence of equivalent forms" (Art. 631), that in Symbolical Algebra, the same expressions continue to be equivalent to each other for *all values* of those indices: or in other words, that

$$a^m \times a^n = a^{m+n},$$

whatever be the values of m and n.

Principle of indices.

This equation, which forms one of the most important propositions in Symbolical Algebra, is sometimes called "the principle of indices." We shall proceed to notice some of the numerous conclusions which are deducible from it, beginning with examples of the interpretation of powers whose indices are fractional or negative numbers.

Interpretation of $a^{\frac{1}{2}}$.

636. What is the meaning of $a^{\frac{1}{2}}$?

The product $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$, (Art. 42) by the "principle of indices" (Art. 635): and it likewise appears that $\sqrt{a} \times \sqrt{a} = a$, where $\sqrt{a}$ denotes the square root of a (Art. 223): we conclude, therefore, that $a^{\frac{1}{2}}$ is identical in meaning with $\sqrt{a}$, inasmuch as when multiplied into itself, it produces the same result*.

Meaning of $a^{\frac{1}{3}}$.

637. What is the meaning of $a^{\frac{1}{3}}$?

The product of $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$: and the product of $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$, where $\sqrt[3]{a}$ is the cube root of a: it follows, therefore, that $a^{\frac{1}{3}}$ is identical in meaning with the cube root of a.

Meaning of $a^{\frac{1}{n}}$.

638. What is the meaning of $a^{\frac{1}{n}}$, where n is a whole number?

If $a^{\frac{1}{n}}$ be successively multiplied into itself, and repeated as a factor n times, the index of its product will be the sum of the indices of the component factors, and therefore equal to n times $\frac{1}{n}$ or 1, and the product itself will be equal to a: the n^{th} root of a or $\sqrt[n]{a}$ possesses the same property, and we conclude, therefore, that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Meaning of $a^{\frac{4}{5}}$ and $a^{\frac{m}{n}}$.

639. What is the meaning of $a^{\frac{4}{5}}$?

* Those quantities which produce the same result, when employed in the same operations, are considered as identical: this principle, though correct in Arithmetical Algebra, will require some modification in Symbolical Algebra: thus $a \times a = a^2 = -a \times -a$: and we should not be justified, in concluding from thence, that $a = -a$. We shall have occasion to notice this ambiguity more at length, when we come to the consideration of multiple values and of the roots of 1 and -1.

The continued product

$$a^{\frac{4}{5}} \times a^{\frac{4}{5}} \times a^{\frac{4}{5}} \times a^{\frac{4}{5}} \times a^{\frac{4}{5}} = a^{5\times\frac{4}{5}} = a^4 = \sqrt[5]{a^4} \times \sqrt[5]{a^4} \times \sqrt[5]{a^4} \times \sqrt[5]{a^4} \times \sqrt[5]{a^4}:$$

we conclude, therefore, that $a^{\frac{4}{5}}$ is identical with $\sqrt[5]{a^4}$, or with the fifth root of the fourth power of a.

More generally, if m and n be whole numbers, it may be shewn in the same manner that $a^{\frac{m}{n}} = \sqrt[n]{a^m}$, or the n^{th} root of the m^{th} power of a.

640. The preceding examples will be sufficient to shew that powers with fractional indices, though not deducible from the definitions of Arithmetical Algebra, will correctly express quantities which are recognized in that science; we shall proceed further to shew that powers with negative indices will likewise admit of an equally simple and consistent interpretation. Negative indices.

If m and n be whole numbers, where m is greater than n, we have shewn (Art. 633), that $a^m \times \frac{1}{a^n} = \frac{a^m}{a^n} = a^{m-n}$: and it follows, from the "general principle of indices" (Art. 633), that $a^m \times a^{-n} = a^{m-n}$. We conclude, therefore, for the particular case under consideration, that Proof that $a^{-n} = \frac{1}{a^n}$.

$$a^{-n} = \frac{1}{a^n}:$$

and we infer, by "the principle of the permanence of equivalent forms," that this proposition is true whatever be the value of n.

It follows, therefore, that

$$a^{-1} = \frac{1}{a}: \quad a^{-2} = \frac{1}{a^2}.$$ Examples.

$$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}: \quad a^{-\frac{1}{3}} = \frac{1}{a^{\frac{1}{3}}}: \quad a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}}.$$

$$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}: \quad a^{-\frac{4}{5}} = \frac{1}{a^{\frac{4}{5}}}: \quad a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}.$$

And, conversely,

$$a = \frac{1}{a^{-1}}: \quad a^2 = \frac{1}{a^{-2}}.$$

$$a^{\frac{1}{2}} = \frac{1}{a^{-\frac{1}{2}}}: \quad a^{\frac{1}{3}} = \frac{1}{a^{-\frac{1}{3}}}: \quad a^{\frac{1}{n}} = \frac{1}{a^{-\frac{1}{n}}}.$$

$$a^{\frac{2}{3}} = \frac{1}{a^{-\frac{2}{3}}}: \quad a^{\frac{4}{5}} = \frac{1}{a^{-\frac{4}{5}}}: \quad a^{\frac{m}{n}} = \frac{1}{a^{-\frac{m}{n}}}.$$

Proof that $a^0=1$.

641. Amongst other important consequences of the preceding proposition, it will follow that $a^0=1$.

For $\frac{a^m}{a^m}=a^{m-m}=a^0$: and since $\frac{a^m}{a^m}=1$, it appears that $a^0=1$, whatever be the value of a.

Proof that $(a^m)^n=a^{mn}$.

642. If n be a whole number, it may be easily shewn that

$$(a^m)^n=a^{mn}.$$

For, in this case, $(a^m)^n$ denotes the continued product of a^m into itself, where a^m is repeated as a factor n times, and the index of the power which is equivalent to this product is equal to the sum of the indices of the component factors (Art. 632), which is mn or n times the index of one of them: and since n and m are general in form, though n is specific in value, it will follow from "the principle of the permanence of equivalent forms" that the proposition is true for all values of m and n whatsoever.

It follows, therefore, that

$$(a^2)^{\frac{1}{2}}=a^{\frac{2}{2}}=a=\sqrt{a^2}\,*.$$

$$(a^m)^{\frac{1}{n}}=a^{\frac{m}{n}}=\sqrt[n]{a^m}.$$

$$(a^{\frac{1}{2}})^{\frac{1}{3}}=a^{\frac{1}{6}}=\sqrt[6]{a}.$$

$$(a^{-\frac{1}{2}})^{-\frac{1}{3}}=a^{\frac{1}{6}}=\sqrt[6]{a}.$$

643. The preceding properties of powers and their indices will enable us not merely to vary to an almost endless extent the equivalent forms of expressions in which they occur, but likewise to reduce them to the most simple forms which they are capable of receiving: the following are examples.

Examples of the reduction of expressions in which indices occur.

(1) $a^2\times\frac{1}{a^3}=a^{2-3}=a^{-1}=\frac{1}{a}$.

(2) $a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{1}{2}+\frac{1}{3}}=a^{\frac{5}{6}}$.

(3) $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}=a^{\frac{1}{2}-\frac{1}{3}}=a^{\frac{1}{6}}=a^{\frac{1}{2}}\times a^{-\frac{1}{3}}$.

(4) $a^{\frac{1}{3}}\times a^{-\frac{1}{4}}=a^{\frac{1}{3}-\frac{1}{4}}=a^{\frac{1}{12}}=\frac{a^{\frac{1}{3}}}{a^{\frac{1}{4}}}$.

* The square root of a^2 may be $-a$ as well as $+a$.

(5) $(a^{\frac{1}{2}})^{\frac{1}{3}} = a^{\frac{1}{6}} = \sqrt[3]{}\sqrt{a}.$

(6) $\{(a^{\frac{1}{2}})^{\frac{1}{3}}\}^{\frac{1}{4}} = a^{\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}} = a^{\frac{1}{24}} = \sqrt[4]{}\sqrt[3]{}\sqrt{a}.$

(7) $(a^{-1})^{-2} = a^2 = \left(\frac{1}{a}\right)^{-2} = \frac{1}{\left(\frac{1}{a}\right)^2}.$

(8) $\{(a^{-1})^{-2}\}^{-3} = a^{-1\times-2\times-3} = a^{-6}.$

(9) $(a^2 b^{-\frac{1}{2}} c^{\frac{2}{5}})^{-\frac{1}{4}} = a^{-\frac{1}{2}} b^{\frac{1}{8}} c^{-\frac{1}{10}} = a^{-\frac{20}{40}} b^{\frac{5}{40}} c^{-\frac{4}{40}}$

(reducing the indices to a common denominator)

$$= \left(\frac{b^5}{a^{20}c^4}\right)^{\frac{1}{40}}.$$

(10) $\{ab^2\sqrt{(ab^3)}\sqrt[3]{(ab^4)}\sqrt[4]{(ab^5)}\}^{\frac{1}{5}} = \{ab^2(ab^3)^{\frac{1}{2}}(ab^4)^{\frac{1}{3}}(ab^5)^{\frac{1}{4}}\}^{\frac{1}{5}}$

$$= \{ab^2 \times a^{\frac{1}{2}}b^{\frac{3}{2}} \times a^{\frac{1}{3}}b^{\frac{4}{3}} \times a^{\frac{1}{4}}b^{\frac{5}{4}}\}^{\frac{1}{5}} = \{a^{\frac{25}{12}}b^{\frac{73}{12}}\}^{\frac{1}{5}} = a^{\frac{5}{12}}b^{\frac{73}{60}}.$$

(11) $\left\{(a+x)\sqrt{\left(\frac{b^3}{(a+x)^{\frac{1}{2}}}\right)}\right\}^{\frac{1}{3}} = \left\{\frac{(a+x)b^{\frac{3}{2}}}{(a+x)^{\frac{1}{4}}}\right\}^{\frac{1}{3}}$

$$= \{b^{\frac{3}{2}}(a+x)^{\frac{3}{4}}\}^{\frac{1}{3}} = b^{\frac{1}{2}}(a+x)^{\frac{1}{4}} = \sqrt{b}\,\sqrt[4]{(a+x)}.$$

In many of the preceding examples, the indication of roots is effected by signs as well as by indices; but, in the process of reduction it is generally convenient and sometimes necessary to replace the signs of the different roots by the corresponding indices: it would in fact conduce very greatly to the uniformity and clearness of algebraical notation, if the use of radical signs was altogether abandoned.

644. The following are miscellaneous Examples of the reduction of expressions involving radical quantities or indices to other and equivalent forms: Other examples of reduction.

(1) $$\frac{\sqrt{(1-x)} + \frac{1}{\sqrt{(1+x)}}}{1 + \frac{1}{\sqrt{(1-x^2)}}} = \sqrt{(1+x)}.$$

For the product of $\sqrt{(1-x)}$ and $\sqrt{(1+x)}$ is $\sqrt{(1-x^2)}$.

$$(2)\quad \frac{\sqrt{(1-x^2)}+\dfrac{x^2}{\sqrt{(1-x^2)}}}{1-x^2}=\frac{1}{(1-x^2)^{\frac{3}{2}}}.$$

We multiply the numerator and denominator by $\sqrt{(1-x^2)}$.

$$(3)\quad \frac{1+\dfrac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{(a^2+x^2)}}.$$

We multiply the numerator and denominator by

$$\sqrt{(a^2+x^2)}.$$

$$(4)\quad \frac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}=a^{\frac{1}{2}}+b^{\frac{1}{2}}.$$

This is involved in the proposition that $a^2-b^2=(a+b)(a-b)$. (Art. 66.)

$$(5)\quad \frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}=a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.\quad \text{(Art. 86. Ex. 16.)}$$

$$(6)\quad \sqrt{ax}+\frac{ax}{a-\sqrt{ax}}=\frac{a\sqrt{x}}{\sqrt{a}-\sqrt{x}}.$$

For $\sqrt{ax}=\dfrac{\sqrt{ax}\,.(a-\sqrt{ax})}{a-\sqrt{ax}}=\dfrac{a\sqrt{ax}-ax}{a-\sqrt{ax}}$, and, therefore,

$$\sqrt{ax}+\frac{ax}{a-\sqrt{ax}}=\frac{a\sqrt{ax}-ax+ax}{a-\sqrt{ax}}=\frac{a\sqrt{ax}}{a-\sqrt{ax}}=\frac{a\sqrt{x}}{\sqrt{a}-\sqrt{x}}.$$

$$(7)\quad \frac{ax}{\sqrt{(a+x)}}-\frac{2ax^2}{(a+x)^{\frac{3}{2}}}+\frac{ax^3}{(a+x)^{\frac{5}{2}}}=\frac{a^3x}{(a+x)^{\frac{5}{2}}}.$$

CHAPTER XVII.

ON THE EXTRACTION OF SQUARE ROOTS IN SYMBOLICAL ALGEBRA: ORIGIN OF AMBIGUOUS ROOTS, AND OF THE SIGN $\sqrt{-1}$.

645. It will follow, from the Rule of Signs, (Art. 569.) that, in Symbolical Algebra, there are always two roots, differing from each other in their sign only, which correspond to the same square: thus a^2 may equally arise from the product $a \times a$ and $-a \times -a$: $(a+b)^2$ may equally arise from the product $(a+b) \times (a+b)$, and $-(a+b) \times -(a+b)$: $(a-b)^2$ may equally arise from the product $(a-b) \times (a-b)$ and $(b-a) \times (b-a)$,* and similarly for all other squares. It follows, therefore, that in passing from the square to the square root, we shall always find *two roots*, which only differ from each other in their sign; but it is the *positive* square root alone which is recognized in Arithmetical Algebra, and which may therefore be called the *arithmetical* root.

There are always two roots, with different signs, which produce the same square.

Arithmetical square root.

646. We have already had occasion to notice these ambiguous square roots in Arithmetical Algebra (Art. 383) in deducing the square roots of squares, such as $x^2 - 2ax + a^2$ and $a^2 - 2ax + x^2$, which are identical in their arithmetical value, though different in the arrangement of their terms. If the relation of the values of the symbols x and a be known, the Rule for the extraction of the square root of these expressions, which is given in Arithmetical Algebra, would require the terms of the square, and therefore of the root to be arranged in the order of their magnitude, and consequently no ambiguity could exist with respect to the *arithmetical* root, which would be $x-a$ if x was greater than a, and $a-x$ if x was less than a: but if the relation of those values be unknown, as where x is an unknown number to be determined from the solution of the equation which leads to the formation of the square, it is uncertain or ambiguous, whether the root be $x-a$ or $a-x$, until that relation is

Occurrence of ambiguous square roots in Arithmetical Algebra.

* For $-(a-b) = b-a$, and therefore $-(a-b) \times -(a-b) = (b-a) \times (b-a)$.

assumed or determined*. In this case, however, the ambiguity of the roots originates in the ambiguity of the problem proposed, and not in the independent use of the signs in Symbolical Algebra.

The symbolical roots derived by the same process as in Arithmetical Algebra. Use of the double sign $\pm$.

647. In extracting the square root, we follow, as in all other operations, the same process both in Symbolical and in Arithmetical Algebra, assuming the proper relation of the symbols: and the negative is at once found from the positive root, by merely changing its sign. It is not unusual, likewise, to denote the *double* root by prefixing the *double* sign $\pm$ to it: thus $\pm a$ means equally $+a$ or $-a$, one or both: $\pm(a-b)$ means equally $a-b$ and $b-a$: and similarly in other cases.

648. The following is the Rule† for extracting the square root in Symbolical Algebra:

Rule for extracting the square root.

Arrange the terms of the square according to some symbol of reference (Art. 576); *obliterate the first term of the square, and make its square root the primary term of the root to be found: divide the first remaining term of the square by double the primary term of the root, making the quotient the second term of the root: add this second term, with its proper sign, to double the primary, to form the divisor: multiply the last term of the root into the divisor, and subtract their product from the remainder of the square: if there be any remainder, repeat the same process, considering the terms already found in the root as constituting the single primary term: and so on continually until there is no remainder, or until the process becomes obviously interminable. The second root is found by changing the sign of all the terms of the first.*

Examples of terminable square roots.

649. The following are Examples in which the process terminates.

* We do not assume, in Arithmetical Algebra, that $x-a$ or $a-x$, are equally the roots of $x^2-2ax+a^2$ when the relation of values of x and a is unknown, but that $x-a$ is always the root of $x^2-2ax+a^2$, and $a-x$ the root of $a^2-2ax+x^2$: and it is only when the relation of values of x and a is unknown, that there is nothing to guide us in the selection of one of those forms of the square in preference to the other.

† The Rule for extracting the square root in Arithmetical Algebra has not been formally stated apart from the corresponding Rule in Arithmetic (Art. 218.)

(1) To extract the square root of $a^2 - 2ab + b^2$.

$$\begin{array}{rll} & \overline{a^2} - 2ab + b^2 \ (a - b: & \text{the second root is} \\ 2a - b) & -2ab + b^2 & -(a-b) \text{ or } b - a. \\ \hline & \cdot \quad \cdot & \end{array}$$

We obliterate a^2 and make its square root a, the primary term of the root: we double a $(2a)$, and we divide $-2ab$ by it: the quotient $-b$ is the second term of the root: we *add* (Art. 547) $-b$ to $2a$, making the divisor $2a - b$: and we subtract the product $(2a - b) \times -b$ from the first remainder $-2ab + b^2$, and there is no second remainder.

(2) To find the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

$$\begin{array}{rll} & \overline{a^4} - 2a^3x + 3a^2x^2 - 2ax^3 + x^4, & \\ 2a^2 - ax) & -2a^3x + 3a^2x^2 - 2ax^3 + x^4 & (a^2 - ax + x^2: \text{ the} \\ & -2a^3x + a^2x^2 & \text{second root is} \\ \hline 2a^2 - 2ax + x^2) & +2a^2x^2 - 2ax^3 + x^4 & -a^2 + ax - x^2. \\ & +2a^2x^2 - 2ax^3 + x^4 & \\ \hline & \cdot \quad \cdot \quad \cdot & \end{array}$$

In forming the second divisor, we consider $a^2 - ax$ as the single primary term of the root, the double of which is $2a^2 - 2ax$: we divide the first term $2a^2x^2$ of the second remainder by the first term $2a^2$ of $2a^2 - 2ax$, and the quotient $+x^2$ is the third term of the root: we add $+x^2$ to $2a^2 - 2ax$ to form the second and final divisor.

(3) To extract the square root of

$$4a^2 + 9b^2 + 16c^2 - 12ab + 16ac - 24bc.$$

Making a the symbol of reference, this expression becomes

$$\begin{array}{rl} & \overline{4a^2} - (12b - 16c)a + 9b^2 - 24bc + 16c^2 \ (2a - (3b - 4c), \\ 4a - (3b - 4c)\} & -(12b - 16c)a + 9b^2 - 24bc + 16c^2 \\ \hline & \cdot \quad \cdot \end{array}$$

We divide $-(12b-16c)\,a$, which is the first term of the first and only remainder, by double the primary term of the root or $4a$: we thus get $-(3b-4c)$, which forms the second term of the root*.

The second root is $-2a+3b-4c$.

Examples of interminable square roots.

650. In the following Examples the process leads to an indefinite series.

(1) To extract the square root of a^2+x^2.

$$\begin{array}{rl}
 & \overline{a^2}+x^2\left(a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}-\cdots\cdots\right. \\[2ex]
2a+\dfrac{x^2}{2a}\Bigg) & x^2+\dfrac{x^4}{4a^2} \\ \hline
2a+\dfrac{x^2}{a}-\dfrac{x^4}{8a^3}\Bigg) & -\dfrac{x^4}{4a^2} \\[2ex]
 & -\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6} \\ \hline
2a+\dfrac{x^2}{a}-\dfrac{x^4}{4a^3}+\dfrac{x^6}{16a^5}\Bigg) & +\dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6} \\[2ex]
 & +\dfrac{x^6}{8a^4}+\dfrac{x^8}{16a^6}-\dfrac{x^{10}}{64a^8}+\dfrac{x^{12}}{256a^{10}} \\ \hline
 & -\dfrac{5x^8}{64a^6}+\dfrac{x^{10}}{64a^8}-\dfrac{x^{12}}{256a^{10}}
\end{array}$$

Inasmuch as the number of terms in the subtrahend is always greater than in the remainder, the process can never terminate. It follows, therefore, that

$$\sqrt{(a^2+x^2)}=(a^2+x^2)^{\frac{1}{2}}=\pm\left\{a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\&c.\right\} \quad (1).$$

If we reverse the order of the terms in the square, we shall find, by a similar process,

$$\sqrt{(x^2+a^2)}=(x^2+a^2)^{\frac{1}{2}}=\pm\left\{x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-\frac{5a^8}{64x^7}+\&c.\right\} \quad (2).$$

* Other examples are

(1) $\sqrt{\left(\dfrac{a^2}{b^2}-2+\dfrac{b^2}{a^2}\right)}=\pm\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$.

(2) $\sqrt{\left(\dfrac{9x^4}{16}-\dfrac{7x^3}{2}+\dfrac{101x^2}{18}-\dfrac{14x}{27}+\dfrac{1}{81}\right)}=\pm\left(\dfrac{3x^2}{4}-\dfrac{7x}{3}+\dfrac{1}{9}\right)$.

The only arrangement of the terms in the square which would be recognized in Arithmetical Algebra, is that in which they follow the order of their magnitude, Art. 646: thus, if a be greater than x, it is the first (1) of these series only, which is *convergent:* if a be less than x, it is the second (2): if this order be reversed, the same series are *divergent*, and no approximation is made to the value of the roots by the aggregation of any number of their terms. (Art. 587.) It will be observed, that in both these series, the signs of the terms after the first are alternately positive and negative. Convergent and divergent series.

(2) To extract the square root of $a^2 - x^2$.

The process followed in the last Example will give us

$$\sqrt{(a^2 - x^2)} = \pm \left\{a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \dots\right\}$$

an indefinite series, where all the terms after the first, are negative.

This series is *convergent* or *divergent*, according as a is greater or less than x: in the first case, both the square and its positive root are arithmetical quantities, and we can approximate indefinitely to the value of the latter by the aggregation of the successive terms of the series*: but in the second case, neither the square nor its roots are recognized in Arithmetical Algebra, and we approximate to no definite arithmetical value, by the aggregation of any number of the terms of the resulting *divergent* series. It is hardly necessary to observe, that the general rule given in Art. 648, is equally applicable to the extraction of the square root of $a^2 - x^2$, both when a is greater and less than x. The square root of $a^2 - x^2$ both when a is greater and less than x.

651. But if a, in the expression $a^2 - x^2$, be less than x, we may replace x^2 by $a^2 + b^2$, which gives us Square roots of negative symbols.

$$a^2 - x^2 = a^2 - (a^2 + b^2) = -b^2,$$

and therefore,

$$\sqrt{(a^2 - x^2)} = \sqrt{-b^2}:$$

* Thus, if $a = 5$ and $x = 1$, we find

$$\sqrt{(25 - 1)} = \sqrt{24} = 5 - .1 - .001 - .00002 - .0000005$$

where the sum of two terms is 4.9, of three terms 4.899, of four terms is 4.89898, of five terms 4.8989795, the last of which differs from the true value of the root by less than $\frac{1}{100000000}$th part of the whole.

and it remains to consider, whether any sign is discoverable, which possesses the requisite symbolical conditions to enable us to express the result of the operation (Art. 632) of extracting the square root, under the circumstances thus indicated.

Discovery of the sign which is competent to express the symbolical square root of $-b^2$.

652. In the first place, the squares $+b^2$ and $-b^2$, express the same magnitude, though affected by different signs, and those signs express affection or quality, and not magnitude (Art. 556): thus, if $+b^2$ be the square contained by the sides $+b$ and $+b$, or $-b$ and $-b$, then $-b^2$ will express the equal square contained by $+b$ and $-b$, or by $-b$ and $+b$. (Art. 593.)

In the second place, the squares b^2 and $-b^2$ are respectively equivalent to the symbolical products of 1 and b^2, and -1 and b^2: for

$$1 \times b^2 = 1b^2 = b^2 \text{ (Art. 30), and } -1 \times b^2 = -1b^2 = -b^2,$$

Again, since the square root of a product is equal to the product of the square roots of its factors, it follows that

$$\sqrt{b^2} = \sqrt{1 \times b^2} = \sqrt{1} \times b = \pm 1 \times b^* = \pm b,$$

and

$$\sqrt{-b^2} = \sqrt{(-1)\,b^2} = \sqrt{-1} \times b = \pm \sqrt{-1} \times b^\dagger = \pm b\sqrt{-1}.$$

This sign is $\sqrt{-1}$.

It thus appears that $\sqrt{-1}$ is the sign, which is determined by the rules of Symbolical Algebra, as characteristic of the square root of a negative symbol: for it is the sign, and the only sign, which satisfies the essential symbolical condition that

$$(b\sqrt{-1})^2 = b^2(\sqrt{-1})^2 = b^2 \times -1 = -b^2.$$

It is usual, but not necessary, to write the sign $\sqrt{-1}$ *after* and not *before*, the symbol or expression which it affects: we thus commonly write $b\sqrt{-1}$, and not $\sqrt{-1} \times b$.

Classification of square roots as arithmetical and unarithmetical, or as positive, negative, and imaginary.

653. The square roots of a^2 are a and $-a$, the first of which is arithmetical: and the square roots of $-a^2$ are $a\sqrt{-1}$ and $-a\sqrt{-1}$, neither of which is arithmetical; but it will be ob-

* For $\sqrt{1} = \pm 1$.

† For $\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1$: and

$$-\sqrt{-1} \times -\sqrt{-1} = (-\sqrt{-1})^2 = (\sqrt{-1})^2 = -1:$$

for the symbolical meaning of $\sqrt{-1}$, or of the square root of -1 is, that, if it be multiplied into itself, it will reproduce -1.

served that the several roots $a, -a, a\sqrt{-1}, -a\sqrt{-1}$ differ merely in their signs, which severally affect the same arithmetical magnitude a. The second of these signs $-$ (or -1, if considered as a symbolical factor of its subject, since

$$-a = -1 \times a)$$

has been shewn to admit of an interpretation consistent with its symbolical conditions, when a denotes a line and other magnitudes (Art. 558): and the signs $\sqrt{-1}$ and $-\sqrt{-1}$ will be shewn in a subsequent Chapter, to admit likewise of an interpretation perfectly consistent with their symbolical conditions, when similarly applied.

In the absence of such interpretations, which have only very recently been recognized in Algebra, it has been usual to designate the quantities denoted by $a\sqrt{-1}$ and $-a\sqrt{-1}$ as *unreal, impossible* or *imaginary*.

654. It appears therefore, that the introduction of the additional sign $\sqrt{-1}$, will enable us to give to the inverse operation of extracting square roots the same general application that we were enabled to give to the inverse operation of subtraction, by the use of the sign $-$, as distinct from its use as a sign of operation: and it will be found generally that the inverse operation of *evolution*, as in the case of cubic, biquadratic and higher roots, will admit of a corresponding generalization, by the use of the cubic, biquadratic and higher roots of 1 and -1: the further consideration of such signs, and of their symbolical properties, will be considered in a subsequent Chapter.

The extraction of higher roots than the square will render the introduction of other signs necessary.

CHAPTER XVIII.

APPLICATIONS OF THE FUNDAMENTAL OPERATIONS OF SYMBOLICAL ALGEBRA TO EXPRESSIONS INVOLVING THE SIGN $\sqrt{-1}$.

655. The rules of Symbolical, are those of Arithmetical, Algebra, extended so as to comprehend every symbolical form, whether affected by the signs $-$, $\sqrt{-1}$, or any other sign, the use of which the unlimited values of the symbols submitted to their operation may render necessary; we have already exemplified their application to symbols affected by the signs $+$ and $-$: it remains to extend them to those which are affected by the sign $\sqrt{-1}$.

Symbols affected by the sign $\sqrt{-1}$, may be further affected, like any other symbols, with the signs $+$ and $-$: but it will be found convenient, in order to avoid confusion and ambiguity, not to consider $\sqrt{-1}$, *when used as a sign*, as carrying with it the double sign $\pm$, but as capable of one value only.

The following are Examples of the application of the fundamental operations of Algebra to expressions involving the sign $\sqrt{-1}$.

Examples. (1) $3a + 2b\sqrt{-1} + 4a + 3b\sqrt{-1} = 7a + 5b\sqrt{-1}$.

In this example, $2b\sqrt{-1}$ and $3b\sqrt{-1}$ are combined into one, in the same manner as other *like* terms.

(2) $4a + 3a\sqrt{-1} + 3a - 2a\sqrt{-1} = 7a + a\sqrt{-1}$.

In this case $4a$ and $3a\sqrt{-1}$ are not *like* terms, but $3a\sqrt{-1}$ and $2a\sqrt{-1}$ are so. The final result may be further reduced to the equivalent form $(7 + \sqrt{-1})a$.

(3) $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$.

(4) $(a + b\sqrt{-1}) - (a - b\sqrt{-1}) = 2b\sqrt{-1}$.

(5) $(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = a + c + (b + d)\sqrt{-1}$.

(6) $a\sqrt{-1} \times b\sqrt{-1} = -ab$.

(7) $a\sqrt{-1} \times -b\sqrt{-1} = ab$.

(8) $(a\sqrt{-1})^2 = -a^2$.

(9) $(a\sqrt{-1})^3 = -a^3\sqrt{-1}$.

(10) $(a\sqrt{-1})^4 = a^4$.

It will be observed that the series

$$1,\ \sqrt{-1},\ (\sqrt{-1})^2,\ (\sqrt{-1})^3,\ (\sqrt{-1})^4,\ (\sqrt{-1})^5,\ (\sqrt{-1})^6,$$

is equivalent to the series

$$1,\ \sqrt{-1},\ -1,\ -\sqrt{-1},\ 1,\ \sqrt{-1},\ -1,\ -\sqrt{-1},$$

and is *periodic*, the same term reappearing in every fourth place: thus the 1st, 5th, 9th terms are 1: the 2nd, 6th, 10th terms are $\sqrt{-1}$: the 3rd, 7th, 11th terms are -1: the 4th, 8th, 12th terms are $-\sqrt{-1}$.

This *periodicity* of the square and higher roots of 1, which is connected with very important theories, will be very fully considered in a subsequent Chapter.

(11) $(a + b\sqrt{-1}) \times c\sqrt{-1} = ac\sqrt{-1} - bc$: for the product is equal to $ac\sqrt{-1} + bc(\sqrt{-1})^2$, and $(\sqrt{-1})^2 = -1$.

(12) $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$: the common process of multiplication applied to this Example, would stand as follows;

$$\begin{array}{l}
a + b\sqrt{-1} \\
a - b\sqrt{-1} \\
\hline
a^2 + ab\sqrt{-1} \\
\qquad - ab\sqrt{-1} + b^2 \\
\hline
\qquad a^2 + b^2 \\
\hline
\end{array}$$

It appears, therefore, that $a^2 + b^2$ is equally resolvible into factors with $a^2 - b^2$ (Art. 66): but such factors, involving the sign $\sqrt{-1}$, are not considered as rational factors.

(13) $(a + b\sqrt{-1})(c + d\sqrt{-1}) = ac - bd + (ad + bc)\sqrt{-1}$.

(14) $(a-b\sqrt{-1})(c-d\sqrt{-1})=ac-bd-(ad+bc)\sqrt{-1}.$

(15) $(a+b\sqrt{-1})^2=a^2-b^2+2ab\sqrt{-1}.$

(16) $(a+b\sqrt{-1})^3=a^3-3ab^2+(3a^2b-b^3)\sqrt{-1}.$

(17) $\sqrt{(-a^2+2ab-b^2)}=\pm(a-b)\sqrt{-1}.$

(18) $\sqrt{(4a^2-12ab\sqrt{-1}-9b^2)}=\pm(2a-3b\sqrt{-1}).$

(19) $\sqrt{\{4a^2-b^2+6bc-9c^2-4(ab-3ac)\sqrt{-1}\}}$

$$=\pm\{2a-(b-3c)\sqrt{-1}\}.$$

(20) $a\sqrt{-b}\times c\sqrt{-d}=-ac\sqrt{bd}$: for $a\sqrt{-b}=a\sqrt{b}\sqrt{-1}$ and $c\sqrt{-d}=c\sqrt{d}\sqrt{-1}$, and, therefore, $a\sqrt{-b}\times c\sqrt{-d}$

$$=ac\sqrt{bd}\,.\,(\sqrt{-1})^2=-ac\sqrt{bd}.$$

In Examples 17, 18, 19, the sign $\sqrt{\ }$ is used to denote an operation to be performed, and the results are in every case ambiguous (Art. 645): but in Example 20, the same sign in the expressions $a\sqrt{-b}$ and $c\sqrt{-d}$ is used to denote one only of the results of the operation of extracting the square roots of $-a^2b$ and $-c^2d$, and is therefore not ambiguous: the other roots are $-a\sqrt{-b}$ and $-c\sqrt{-d}$, and are not required to be considered in this Example.

(21) $$\frac{1}{a+b\sqrt{-1}}+\frac{1}{a-b\sqrt{-1}}=\frac{2a}{a^2+b^2}.$$

(22) $$\frac{1}{a-b\sqrt{-1}}-\frac{1}{a+b\sqrt{-1}}=\frac{2b\sqrt{-1}}{a^2+b^2}.$$

(23) $$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}+\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\frac{2(a^2-b^2)}{a^2+b^2}.$$

(24) $$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}-\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\frac{4ab\sqrt{-1}}{a^2+b^2}.$$

(25) $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}+\frac{a-b\sqrt{-1}}{c-d\sqrt{-1}}=\frac{2(ac+bd)}{c^2+d^2}.$$

CHAPTER XIX.

ON THE GENERAL THEORY AND SOLUTION OF QUADRATIC EQUATIONS.

656. THE process of extracting the square root of a number or expression q, is equivalent to the solution of the binomial quadratic equation (Arts. 246 and 377) Binomial quadratic equations.

$$x^2 = q, \text{ or } x^2 - q = 0.$$

For the value of x, determined from this equation is $\sqrt{q}$ or the square root of q: and this root, as we have already shewn, (Art. 645) possesses two values which differ from each other in their sign only: thus, if a represents one root, $-a$ represents the other: and if q, whose root is required, is negative, then $a\sqrt{-1}$ represents one of its two roots, and $-a\sqrt{-1}$ the other. Their double roots.

Thus, in the equation Examples.

$$x^2 - 4 = 0,$$

the roots are 2 and -2: and in the equation

$$x^2 + 9 = 0,$$

the roots are $3\sqrt{-1}$, and $-3\sqrt{-1}$, which are both of them *impossible* or *imaginary* *. (Art. 652.)

657. More generally, the square whose root is required, may present itself in connection with another term which involves the simple root, as in the following forms (Art. 383): Trinomial quadratic equations.

$$x^2 - px = q \qquad (1)$$
$$x^2 + px = q \qquad (2)$$
$$x^2 - px = -q \qquad (3)$$
$$x^2 + px = -q \qquad (4).$$

Or, transposing the significant terms to one side of the equation, these forms become

* They are both of them *unarithmetical* roots, and will generally indicate that the problem in which they originate, is incapable of solution in the sense in which it was proposed.

$$x^2 - px - q = 0 \qquad (5)$$

$$x^2 + px - q = 0 \qquad (6)$$

$$x^2 - px + q = 0 \qquad (7)$$

$$x^2 + px + q = 0 \qquad (8).$$

The three first of these equations have been considered in Art. 384: the last belongs exclusively to Symbolical Algebra.

It is obvious that these four equations are convertible with each other, by merely changing the signs of p and q: and it will follow (for the same reason) that the symbolical solution of one of them will furnish the symbolical solution of all the others.

General solution of quadratic equations.

658. Let us proceed to determine the solution of the equation

$$x^2 - px - q = 0 \qquad (5),$$

$$\text{or } x^2 - px = q.$$

$$\text{Since } \left(x - \frac{p}{2}\right)^2 = x^2 - px + \frac{p^2}{4},$$

and since $x^2 - px = q$, it follows that

$$\left(x - \frac{p}{2}\right)^2 = q + \frac{p^2}{4},$$

and, therefore, (Art. 645)

$$x - \frac{p}{2} = \pm \sqrt{\left\{q + \frac{p^2}{4}\right\}},$$

$$\text{and } x = \frac{p}{2} \pm \sqrt{\left(q + \frac{p^2}{4}\right)}. \qquad (9).$$

If we change the sign of p and retain that of q, as in equation (6), we get

$$x = -\frac{p}{2} \pm \sqrt{\left(q + \frac{p^2}{4}\right)}. \qquad (10).$$

If we retain the sign of p and change that of q, as in equation (7), we get

$$x = \frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} - q\right)}. \qquad (11).$$

If we change the signs both of p and q, as in equation (8), we get

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} - q\right)}^*. \qquad (12).$$

* This process is, in every respect, similar to that which is followed in extracting the square root: thus, if it be required to find the square root of

$$x^2 - px - q,$$

659. It appears from the formulæ (9) and (10) of solution of the equations

In equations (5) and (6) one root is positive and the other negative.

$$x^2 - px - q = 0 \qquad (5)$$
$$x^2 + px - q = 0 \qquad (6)$$

that when p and q are arithmetical quantities, there are two roots, one positive and the other negative, or one arithmetical and the other not: for $\sqrt{\left(q + \frac{p^2}{4}\right)}$ is always greater than $\frac{p}{2}$, and, therefore, $\sqrt{\left(q + \frac{p^2}{4}\right)} \pm \frac{p}{2}$ is always positive, and

$$-\sqrt{\left(q + \frac{p^2}{4}\right)} \pm \frac{p}{2}$$

is always negative.

Examples.

Thus in the equation

$$x^2 - x - 90 = 0,$$

we find $\left(x - \frac{1}{2}\right)^2 = 90 + \frac{1}{4} = \frac{361}{4}$,

and, therefore, $x - \frac{1}{2} = \pm \frac{19}{2}$,

and $x = 10$ or -9.

the process will stand as follows:

$$\begin{array}{r|l} & \overline{x^2} - px - q \quad \left\{ x - \frac{p}{2} \right. \\ 2x - \frac{p}{2} \Big\} & -px + \frac{p^2}{4} \\ \hline & -\frac{p^2}{4} - q \end{array}$$

It follows, therefore, that $\left(x - \frac{p}{2}\right)^2$ differs from $x^2 - px - q$ by $-\frac{p^2}{4} - q$, and, consequently, if $x^2 - px - q = 0$, which the conditions of the equation require, we have $\left(x - \frac{p}{2}\right)^2 = \frac{p^2}{4} + q$,

and, therefore, $x - \frac{p}{2} = \pm \sqrt{\left(\frac{p^2}{4} + q\right)}$,

or $x = \frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)}$.

In the following Chapter, examples will be given of biquadratic and higher equations of an even order, which admit of reduction or solution, through the ordinary processes for extracting roots.

In the equation

$$x^2 + 4x - 21 = 0,$$

$$\text{we find } (x + 2)^2 = 21 + 4 = 25,$$

$$\text{and, therefore, } x + 2 = \pm 5,$$

$$\text{and } x = 3 \text{ or } - 7.$$

It is the positive root *alone* which is recognized in Arithmetical Algebra.

In equation (7), both roots are possible or both impossible: in equation (8) both roots are negative or both impossible.

660. Again, it appears from the formulæ (11) and (12) of solution of the equations

$$x^2 - px + q = 0 \qquad (7)$$

$$x^2 + px + q = 0 \qquad (8),$$

that, when p and q are arithmetical quantities, and $\frac{p^2}{4}$ *greater* than q, there are two positive roots of the first equation (7), and two negative roots of the second (8): for $\sqrt{\left(\frac{p^2}{q} - q\right)}$ is always less than $\frac{p}{2}$, and therefore the signs of the two roots are determined by the sign of $\frac{p}{2}$.

Thus in the equation

$$x^2 - 7x + 12 = 0,$$

$$\text{we find } \left(x - \frac{7}{2}\right)^2 = \frac{49}{4} - 12 = \frac{1}{4},$$

$$\text{and, therefore, } x - \frac{7}{2} = \pm \frac{1}{2},$$

$$\text{and } x = 4 \text{ or } 3.$$

In other words, the solution is *ambiguous*.

In the equation

$$x^2 + 12x + 35 = 0,$$

$$\text{we find } (x + 6)^2 = 36 - 35 = 1,$$

$$\text{and, therefore, } x + 6 = \pm 1,$$

$$\text{and } x = - 7 \text{ or } - 5.$$

This solution is arithmetically impossible: but if negative values be recognized as admitting of interpretation, the solution is also ambiguous like the one preceding.

If, in the same formulæ (11) and (12), $\frac{p^2}{4}$ is less than q, then $\frac{p^2}{4} - q$ is negative, and the roots of equations (7) and (8) will involve the square root of a negative quantity and therefore the sign $\sqrt{-1}$. (Art. 652.)

Thus in the equation

$$x^2 - 8x + 25 = 0,$$

$$\text{we find } (x-4)^2 = 16 - 25 = -9,$$

$$\text{and, therefore, } x - 4 = \pm 3\sqrt{-1},$$

$$\text{and } x = 4 + 3\sqrt{-1}, \text{ or } 4 - 3\sqrt{-1}.$$

Again, in the equation

$$x^2 + 3x + \frac{97}{36} = 0,$$

$$\text{we find } \left(x + \frac{3}{2}\right)^2 = \frac{9}{4} - \frac{97}{36} = -\frac{4}{9},$$

$$\text{and, therefore, } x + \frac{3}{2} = \pm \frac{2}{3}\sqrt{-1},$$

$$\text{and } x = -\frac{3}{2} + \frac{2}{3}\sqrt{-1}, \text{ or } -\frac{3}{2} - \frac{2}{3}\sqrt{-1}.$$

The unarithmetical or *impossible* roots which present themselves in equations (7) and (8), necessarily assume the form $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, where the term involving the sign $\sqrt{-1}$ is affected with the sign + in one of them, and with the sign − in the other.

661. The roots, which are thus determined, bear a very simple relation to the *given* quantities of the equations to which they belong; *their symbolical sum being always equal to the coefficient of the second term with its sign changed, and their product to the last term:* thus, if a and b be the roots of the equation $x^2 - px + q = 0$, we find

Composition of the coefficient of the second term and of the last term, in terms of the roots.

$$a + b = \left\{\frac{p}{2} + \sqrt{\left(\frac{p^2}{4} - q\right)}\right\} + \left\{\frac{p}{2} - \sqrt{\left(\frac{p^2}{4} - q\right)}\right\} = p,$$

and

$$ab = \left\{\frac{p}{2} + \sqrt{\left(\frac{p^2}{4} - q\right)}\right\} \times \left\{\frac{p}{2} - \sqrt{\left(\frac{p^2}{4} - q\right)}\right\} = \frac{p^2}{4} - \left(\frac{p^2}{4} - q\right) = q.$$

A change of sign in p and q, one or both, will not affect this conclusion, it being assumed that a and b may be either positive, negative, one or both, or both of them imaginary.

Resolution of a quadratic trinomial into its simple factors.

662. The same assumptions being made, it may be further shewn that $x-a$ and $x-b$ are the simple factors of the trinomial

$$x^2 - px + q.$$

For $x - a = x - \frac{p}{2} - \sqrt{\left(\frac{p^2}{4} - q\right)}$,

$$x - b = x - \frac{p}{2} + \sqrt{\left(\frac{p^2}{4} - q\right)},$$

and, therefore,

$$(x-a)(x-b) = \left(x - \frac{p}{2}\right)^2 - \left(\frac{p^2}{4} - q\right) = x^2 - px + \frac{p^2}{4} - \frac{p^2}{4} + q$$
$$= x^2 - px + q.$$

It will follow, therefore, that $x^2 - px + q$ will become equal to zero, when one of its factors is equal to zero, or when $x = a$, or $x = b$, and in *no other case:* in other words, a and b are the only values of x which verify the equation

$$x^2 - px + q = 0^*.$$

663. The propositions contained in the two last articles, will be found to be included in some general propositions which apply to the coefficients of equations of all dimensions, when they assume the ordinary form.

Examples of the solution of quadratic equations.

664. The following examples, of the solution of quadratic equations, and of problems which lead to them, may be considered as supplementary to those which are given in Arts. 389, and 410, and 412†.

* It may be otherwise shewn, that, if a be a value of x, which makes $x^2 - px + q = 0$, then $x - a$ is a factor of the trinomial $x^2 - px + q$ for all values of x: for by the assumption it appears that $a^2 - pa + q = 0$, and consequently

$$x^2 - px + q = x^2 - px + q - (a^2 - pa + q) = x^2 - a^2 - p(x-a):$$

$$\text{and, therefore, } \frac{x^2 - px + q}{x - a} = x + a - p = x - (p - a):$$

it follows, therefore, that $x - a$ is a factor of $x^2 - px + q$: the other factor is $x - (p - a)$.

† In Example 1, Art. 389, the second or negative root is -3, and in Example 3, it is -14: the two roots in Example 5 are both negative, one being $-\frac{1}{8}$, and the other -4: the solutions in Examples 2 and 4 are ambiguous, and both the

(1) Let $\frac{10}{x} - \frac{3}{x+2} = \frac{10}{x+1}$. Examples.

Clearing the equation of fractions, Art. 368, we get

$$10x^2 + 30x + 20 - 3x^2 - 3x = 10x^2 + 20x.$$

Collecting like terms into one (Art. 372), and transposing the known terms to one side of the equation, and the unknown to the other, we find $3x^2 - 7x = 20$.

Dividing both sides by 3 (Art. 370), which is the coefficient of x^2, we get $x^2 - \frac{7x}{3} = \frac{20}{3}$.

Completing the square (Arts. 384 and 657), the equation becomes $\left(x - \frac{7}{6}\right)^2 = \frac{49}{36} + \frac{20}{3} = \frac{289}{36}$.

Extracting the square root, we get

$$x - \frac{7}{6} = \pm \frac{17}{6},$$

and therefore

$$x = 4, \text{ or } -\frac{5}{3}.$$

There is only one arithmetical root: but the negative root $-\frac{5}{3}$ fully satisfies, as may be found by trial, the symbolical conditions of the equation.

(2) Let $\frac{x}{7-x} + \frac{7-x}{x} = \frac{29}{10}$.

The several processes of reduction and solution follow the same order as in the last example.

$$10x^2 + 490 - 140x + 10x^2 = 203x - 29x^2,$$

$$49x^2 - 343x = -490,$$

$$x^2 - 7x = -10,$$

$$\left(x - \frac{7}{2}\right)^2 = \frac{49}{4} - 10 = \frac{9}{4},$$

$$x - \frac{7}{2} = \pm \frac{3}{2},$$

$$x = 5, \text{ or } 2.$$

roots are determined by the processes of Arithmetical Algebra. In Example 5, Art. 410, the second or negative root is -9: in Example 6, it is -15: in Example 4, Art. 412, it is -15.

This is an example of an equation, whose roots are arithmetically ambiguous.

(3) Let $\frac{3x+25}{4} - \frac{7}{21+2x} - x = 7$,

$$63x + 525 + 6x^2 + 50x - 28 - 84x - 8x^2 = 588 + 56x,$$

$$2x^2 + 27x = -91,$$

$$x^2 + \frac{27x}{2} = -\frac{91}{2},$$

$$\left(x + \frac{27}{4}\right)^2 = \frac{729}{16} - \frac{91}{2} = \frac{1}{16},$$

$$x + \frac{27}{4} = \pm\frac{1}{4},$$

$$x = -7, \text{ or } -\frac{13}{2}.$$

This equation admits of no arithmetical root, but its symbolical conditions are satisfied by both the negative roots -7 and $-\frac{13}{2}$.

(4) Let $\frac{x^2-4x}{13} + \frac{3x^2+11}{4} + 8 = 3x$,

$$4x^2 - 16x + 39x^2 + 143 + 416 = 156x,$$

$$43x^2 - 172x = -559,$$

$$x^2 - 4x = -\frac{559}{43} = -13,$$

$$x^2 - 4x + 4 = -9,$$

$$x - 2 = \pm 3\sqrt{-1},$$

$$x = 2 \pm 3\sqrt{-1}.$$

There is no arithmetical or negative root which will satisfy the equation: the two roots are said to be imaginary*.

Problems producing quadratic equations and interpretation of their results.

665. We shall endeavour to connect the solution of the following problems producing quadratic equations, with the illustration of some of the more common principles of interpreting the results of symbolical operations.

* In the examples of quadratic equations which are given in the Note, Art. 389, the Symbolical, as distinguished from the Arithmetical, roots, are in Example 1, -10: in Example 2, -7: in Example 4, $-\frac{67}{11}$: in Example 5, -1: in Example 6, -10: in Example 8, $-\frac{5}{3}$: in Example 11, $2+\sqrt{-2}$, or $2-\sqrt{-2}$.

(1) To divide the number 10 into two parts, whose product shall be equal to 24.

Let x be one of the two parts, and therefore $10-x$ the other. Their product $=x(10-x)=24$: or $10x-x^2=24$,

$$\text{and, therefore, } x^2-10x=-24.$$

Completing the square, we get $(x-5)^2=25-24=1$,

$$\text{and, therefore, } x-5=\pm 1,$$

$$\text{and } x=6, \text{ or } 4.$$

The two parts into which 10 is divided are 6 and 4, and there is no ambiguity in the solution of the problem: but the value of x is ambiguous, inasmuch as, conformably to the assumptions which are made, it may equally represent the greater or the lesser of the two parts into which 10 is divided.

If the product had been 25 instead of 24, we should have found

$$x^2-10x=-25,$$
$$(x-5)^2=25-25=0,$$
$$x-5=0,$$
$$x=5.$$

In this case 10 would have been divided into two equal parts*, and there would have been no ambiguity either in the solution of the problem or in the value of x.

If the product was required to be 26 instead of 24, the equation would have been

$$x^2-10x=-26,$$
$$(x-5)^2=25-26=-1,$$
$$x-5=\pm\sqrt{-1},$$
$$x=5\pm\sqrt{-1}.$$

In this case, the solution of the problem is *impossible* in the sense in which it was proposed, and we transfer the same epithet to its symbolical roots: but those roots will be found to satisfy the symbolical conditions of the problem; for their *sum*

$$=(5+\sqrt{-1})+(5-\sqrt{-1})=10,$$

* The product of the parts into which a number is divided, will be the greatest when those parts are equal: for if $2n$ be the number, and $n+x$ one of those parts, and, therefore, $n-x$ the other, their product is $(n+x)(n-x)$, or n^2-x^2, which is the greatest possible when $x=0$.

and their product

$$= (5 + \sqrt{-1}) \times (5 - \sqrt{-1}) = 25 + 1 = 26^*.$$

(2) The sum of a decreasing arithmetical series is 75, its first term 21, and the common difference 3: to find the number of its terms.

The formula in Art. 422, gives us

$$75 = \{42 - 3(n-1)\} \frac{n}{2},$$

where n, or the unknown number, is the number of terms of the series.

We thence get

$$15n - n^2 = 50,$$

$$n^2 - 15n = -50,$$

$$\left(n - \frac{15}{2}\right)^2 = \frac{225}{4} - 50 = \frac{25}{4},$$

$$n - \frac{15}{2} = \pm \frac{5}{2},$$

$$n = 10 \text{ or } 5.$$

If we take 5, the less of these two values, we get the series of five terms

$$21,\ 18,\ 15,\ 12,\ 9,$$

which satisfies the proposed conditions.

If we take 10, or the greater of the two values of n to express the number of terms, the series will be found to be

$$21,\ 18,\ 15,\ 12,\ 9,\ 6,\ 3,\ 0,\ -3,\ -6,$$

which likewise answers the conditions of the problem, inasmuch as the symbolical sum of the 5 last terms $6, 3, 0, -3, -6$, is equal to *zero*: but the arithmetical solution of the problem, as proposed, is not ambiguous, inasmuch as it is not possible to form 10 consecutive terms of the decreasing series 21, 18, 15, ... without introducing, and therefore recognizing, negative terms.

* Another equation of the same class is given in Example 7, Art. 410: the student is particularly recommended to study the problems leading to quadratic equations, which are given in that and the following articles, with especial reference to the more enlarged views of the interpretation of symbolical results which are given in this Chapter.

The arithmetical conditions, however, of the equation

$$75 = \{42 - 3(n-1)\}\frac{n}{2},$$

to which the problem leads, are equally satisfied by $n = 5$ and $n = 10$: for the first gives

$$75 = (42 - 12) \times \frac{5}{2},$$

and the second,

$$75 = (42 - 27) \times 5,$$

which are severally equal to each other.

(3) Given one line, to find another such, that the rectangle which they form, shall be equal to the square of their difference. A geometrical problem stated generally.

Let a be the given line, and x the line to be determined: then by the conditions of the problem, we form the equations

$$a x = (x - a)^2 \text{ or } (a - x)^2 \qquad (1),$$

which equally lead to the equation

$$x^2 - 3ax = -a^2.$$

Completing the square,

$$x^2 - 3ax + \frac{9a^2}{4} = \frac{5a^2}{4},$$

$$x - \frac{3a}{2} = \pm\frac{a\sqrt{5}}{2},$$

$$x = \frac{3a}{2} \pm \frac{a\sqrt{5}}{2},$$

$$a + \frac{1+\sqrt{5}}{2}a,$$

$$a + \frac{1-\sqrt{5}}{2}a.$$

Both these roots are positive, since $\sqrt{5} = 2.236\ldots$, and, therefore, $\dfrac{1+\sqrt{5}}{2} = 1.618$ and $\dfrac{1-\sqrt{5}}{2} = -.618\ldots$; consequently $x = 2.618a$, or $.382a$, nearly.

In the problem proposed, it is uncertain whether the line to be determined is less or greater than the given line: and the symbolical solution, being coextensive with the enunciation of the problem, includes both cases. Its solution includes two cases.

The first case is equivalent to the following problem. Its first case stated and solved.

"To divide a given line into two such parts that the rectangle contained by the whole and one of the parts shall be equal to the square of the other."

If we call x the greater of the two parts, and, therefore, $a - x$ the other, the equation will become

$$a(a - x) = x^2 \qquad (2),$$

the symbolical roots of which are

$$\frac{(\sqrt{5} - 1)}{2} a, \text{ and } - \frac{(\sqrt{5} + 1)}{2} a,$$

one being positive and the other negative: the solution is therefore not ambiguous, there being only one positive root. But if x be taken to represent the less of the two parts into which the line is divided, the equation becomes

$$ax = (a - x)^2,$$

the solution of which, as we have seen above, is ambiguous: this ambiguity, however, is removed, by the necessity of taking the least of the values of x or $.382a$, as the only value which is compatible with the conditions of the problem.

Its second case stated and solved.

The second case corresponds to the following problem.

"To find a point in a given line produced, such that the rectangle formed by the given line and the whole line produced, shall be equal to the square of the part of it produced."

If we call x the part produced, and therefore $a + x$ the whole line produced, the conditions of the problem lead to the equation

$$a(a + x) = x^2 \qquad (3),$$

the two roots of which are $\frac{(\sqrt{5} + 1)}{2} a$, and $- \frac{(\sqrt{5} - 1)}{2} a$, differing merely in sign from those of equation (2): it is the arithmetical root alone which corresponds to the problem: but if we had called x the whole line produced, the resulting equation would have been

$$ax = (x - a)^2,$$

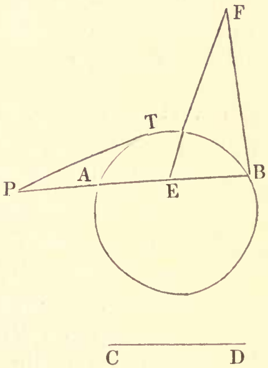
the solution of which is ambiguous, though the problem is not, the less of the two roots being excluded by the conditions which it imposes.

It thus appears that the same problem may lead to one equation whose solution is unambiguous, and to another which is ambiguous: but in the latter case, the conditions of the pro-

blem will immediately lead to the exclusion of one of the two roots, and therefore to the removal of the ambiguity.

Whatever geometrical problems can be shewn to depend immediately or ultimately upon the division of a line into extreme and mean ratio, will lead likewise to some one of the equations which we have just been considering*.

(4) To find a point in a given chord of a circle produced from whence the tangent drawn to the circle shall be equal to a given line. A geometrical problem.



Let $AB\,(2c)$ be the given chord, and $CD\,(l)$ the given line: *assume* P to be the point in the chord BA produced, from which the tangent PT drawn to the circle is equal to CD: and let PA the part of the chord produced and which is required to be determined, be denoted by x.

Then, by the property of the circle (Euclid, Book III. Prop. 36.) we get

$$PA \times PB = PT^2;$$

or replacing PA, PB and PT, by x, $x+2c$ and l,

$$x(x+2c) = l^2,$$

$$\text{or } x^2 + 2cx = l^2 \qquad (1).$$

Completing the square, we get

$$(x+c)^2 = c^2 + l^2,$$

$$\text{and } x + c = \pm \sqrt{(c^2+l^2)},$$

$$\text{or } x = \sqrt{(c^2+l^2)} - c \text{ and } -\sqrt{(c^2+l^2)} - c.$$

It is the first of these values *only* which belongs to Arithmetical Algebra, and which corresponds to the problem proposed.

But if we had denoted PB and not PA, by x, we should have found the equation

$$x^2 - 2cx = l^2 \qquad (2),$$

the positive root of which, or $c + \sqrt{(c^2+l^2)}$ gives the value of PB required, and is identical in *magnitude* with the negative root of the first equation (1).

In the algebraical solution of this and other problems, we begin by *assuming the solution* of the problem to be solved The analysis of this problem.

* Such is the problem for constructing a triangle where each angle at the base is double of the angle at the vertex, (Euclid, Book IV. Prop. X.) upon which the inscription of a regular pentagon in a circle depends, as well as others which might be proposed.

and we subsequently determine the quantity whose value is required, by following out the necessary consequences of the conditions which it must satisfy until we arrive at one which can be made the basis of an equation: thus, in the first solution of the problem under consideration, we *assume* P to be the point in the chord $BA\,(2c)$ produced, from which the tangent PT, drawn to the circle, is equal in length to the given line $CD\,(l)$, and we denote PA, the assumed distance of P from A, by x: the conditions of the problem, which are involved in the general property of the tangent of a circle, shew that x must have such a value that

$$x\,(x+2c)=l^2:$$

Its synthesis. and it is found that the only arithmetical value of x, which will satisfy this equation, is $\sqrt{(c^2+l^2)}-c$.

The preceding process of investigation may be called the *analysis* of the problem proposed, and immediately leads to its *synthesis* or construction, which is as follows.

Bisect the chord BA in E: draw BF perpendicular to AB and equal to CD or l: join EF, which is equal to

$$\sqrt{(EB^2+BF^2)}=\sqrt{(c^2+l^2)}=c+x=EP:$$

it follows, therefore, that a circle described from the centre E, with the radius EF, will cut BA produced in the point P required.

Geometrical analysis and synthesis. The corresponding process in Geometry, by which, assuming the problem to be solved and the unknown line or other quantity to be determined, we are enabled to discover, from an examination of the conditions which it must satisfy or the properties it must possess, the direct means of determining its value, is likewise called its *analysis:* whilst the inverse process, by which, when those consequences are traced out or discovered, we are enabled to *construct* or *solve* the problem, is called its *synthesis*. Thus, in the problem under consideration, we begin by *assuming* the point P to be that from which the tangent PT drawn to the circle is equal to CD: we further know, from the property of the circle, that

$$PA\times PB=PT^2:$$

we likewise know, if we bisect AB, that $PA\times PB+AE^2$ is equal to PE^2 (Euclid, Book II. Prop. 6), and, therefore, to PT^2+AE^2 or to CD^2+AE^2, (since $PT^2=CD^2$), or to the hypothenuse (EF) of a right-angled triangle EBF, of which EB and BF (which is equal to CD), are the sides: if, therefore, CD and AE be given, the distance EP of the required point from the given point E is also given.

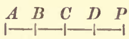
This is the *geometrical analysis* of the problem: its *synthesis,* which is the ordinary form of *exposition* adopted in Geometry, directs us to bisect BA in E: to draw BF perpendicular to EB and equal to CD: to join EF: and in BA produced to take EP equal to EF: and we finally assert that the tangent PT drawn to the circle from the point P thus determined, is equal to the given line CD. We then subjoin the proof, by which the correctness of this construction, or *synthesis,* is established.

The processes of Algebra are analytical.

The processes of exposition in Geometry are generally *synthetical,* whilst those of discovery in that science, and of discovery and of exposition in Algebra, are uniformly *analytical:* it is this fundamental distinction in the ordinary mode of exhibiting or deducing the conclusions in Geometry and Algebra, which has led to the very general application of the term *analytical* to algebraical processes, as distinguished from those which are geometrical, though the processes of discovery in the two sciences are essentially the same.

A geometrical problem.

666. Problem. Given four points in the same straight line, to find a fifth, such that the rectangle under its distances from the first and second may bear a given ratio to the rectangle under its distances from the third and fourth.



Let A, B, C and D be the given points, and let P be the point required: let $AD = a$, $BD = b$: $CD = c$ and $DP = x$, the lines being estimated from A towards P (Art. 558); it will follow, therefore, that $CP = x + c$, $BP = x + b$ and $AP = x + a$: and the conditions of the problem give us

$$\frac{PA \times PB}{PC \times PD} = \frac{(x+a)(x+b)}{(x+c)\,x} = 1 + e \qquad (1),$$

where e is a given number, and which leads, when reduced, to the equation

$$ex^2 - \{a + b - (1+e)\,c\}\,x = a\,b \qquad (2),$$

$$\text{or } x^2 - \left(\frac{a+b-c}{e} - c\right)x = \frac{a\,b}{e} \qquad (3).$$

The solution of this equation gives us

$$x = \tfrac{1}{2}\left(\frac{a+b-c}{e} - c\right) \pm \tfrac{1}{2}\sqrt{\left\{\left(\frac{a+b-c}{e} - c\right)^2 + \frac{4ab}{e}\right\}} \qquad (4).$$

Discussion of its solution.

If e be positive and if both a and b have the same sign, or *be drawn in the same direction,* the two roots are of the form α and $-\beta$ (Art. 659), and shew that there are two points on *different* sides of the point D, whose values those roots determine, and which equally satisfy the conditions of the problem.

Limiting values.

If e diminishes, the value of a or of the *positive* root increases and becomes *indefinitely* great when e is *indefinitely* small: the *negative* root, under the same circumstances, becomes ultimately equal to $\dfrac{-ab}{a+b-c}$ *.

If, in equation (1), we replace $1+e$ by $1-e$, the values of x will be expressed by

$$x = -\tfrac{1}{2}\left(\frac{a+b-c}{e}+c\right) \pm \tfrac{1}{2}\sqrt{\left\{\left(\frac{a+b-c}{e}+c\right)^2 - \frac{4ab}{e}\right\}},$$

which are both of them negative, shewing that there are two points on the left of the point D, which answer the conditions of the problem: but if we reverse the directions, and, therefore, the signs of a, b, c and x, the two roots, under the same circumstances, will be both of them positive†.

* For the equation (2), if e become evanescent, degenerates into the simple equation $-(a+b-c)x = ab$, which gives the value of x in the text: the same result may be otherwise obtained from the general expression for x (4): for if we take the negative sign of the square root, and make $t = \dfrac{a+b-c}{e} - c$, we get

$$x = \tfrac{1}{2}t - \tfrac{1}{2}\sqrt{\left(t^2 + \frac{4ab}{e}\right)} = \tfrac{1}{2}t - \tfrac{1}{2}t\sqrt{\left(1+\frac{4ab}{et^2}\right)}:$$

extracting the square root of $1+\dfrac{4ab}{et^2}$ by the process given in Art. 650, Ex. 1, we get

$$x = \tfrac{1}{2}t - \tfrac{1}{2}t\left(1 + \frac{2ab}{et^2} - \frac{2a^2b^2}{e^2t^4} + \&c.\right)$$

$$= \tfrac{1}{2}t - \tfrac{1}{2}t - \frac{ab}{et} + \frac{a^2b^2}{e^2t^3} - \&c.$$

$$= -\frac{ab}{et} + \frac{a^2b^2}{e^2t^3} - \&c. \ldots\ldots\ldots\ldots$$

But $et = e\left(\dfrac{a+b-c}{e} - c\right) = a+b-c-ec$, and

$$e^2t^3 = (a+b-c-ec)^2 \times \left(\frac{a+b-c}{e} - c\right):$$

and if e becomes indefinitely small, then et becomes $a+b-c$, and e^2t^3 is infinitely great: it follows therefore, that, under such circumstances,

$$x = -\frac{ab}{et} = -\frac{ab}{a+b-c},$$

all the other terms of the series becoming indefinitely small: the other root of the equation (2), as we have already shewn, is indefinitely great.

† The following are particular examples of this problem.

(1) Let $a=8$, $b=6$, and $c=3$, and let $\dfrac{(x+8)(x+6)}{(x+3)x} = 2$.

The problem of the Determinate Section.

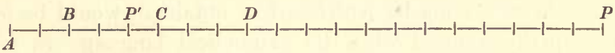
The problem, whose solution we have been considering, was known to the ancient geometers under the name of the *Determinate Section*, and is said to have branched out, with its kindred problems, into eighty-seven propositions*: it would be difficult to select an example better calculated to exhibit the superior brevity and comprehensiveness of symbolical processes.

Process by which the conclusions of geometry are generalized.

The magnitudes, which are the subjects of our reasoning in Geometry, are either *given* and exhibited to the eye, or are required to be *determined* and exhibited: and the conclusions which we draw concerning them, are in the first instance, confined to the specific magnitudes under consideration: it is only by a deductive process that we are enabled to make our conclusions respecting them general, by shewing that the form of the demonstration would remain unchanged, when applied to other magnitudes of the same kind, which present precisely the same conditions.

Thus, if two triangles, which are given or exhibited, have two sides about two equal angles, equal to each other, they are shewn to have their remaining side, angles, and also their areas equal to each other: and the demonstration, and therefore the

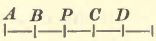
The values of x are $12\,(DP)$, or $-4\,(DP')$.



If we change the signs of a, b, c, and therefore the directions of the lines they represent, we merely change the signs, and therefore the directions of the values of x.

(2) Let $a = 4$, $b = 3$, and $c = 1$, and let $\dfrac{(x+4)(x+3)}{(x+1)x} = 1$.

The only value of x is -2: or P is the middle point between B and C



(3) Let $a = 4$, $b = 3$, $c = 1$, and let $\dfrac{(x+4)(x+3)}{(x+1)x} = \tfrac{1}{2}$.

The values of x are -10.772 and -2.228 nearly.

If we change the signs of a, b and c, or the directions in which they are estimated, we shall change the signs of the values of x, and therefore the directions in which they are reckoned.

* It formed one of the lost treatises of the celebrated Apollonius of Perga, which Pappus has noticed in his Proemium to the 7th book of his Collections, and which Schooten and Robert Simson have succeeded generally in restoring: its object is stated by him in the following terms. "To cut an indefinite straight line in a point, so, that of the lines intercepted between that and other given points in it, the square of one, or the rectangle under two, of them, may bear a given ratio either to the rectangle formed by a given line and another of these intercepted lines, or to the rectangle formed by two of them."

conclusion, being obviously independent of the specific form or values of the triangles, or of their sides and angles, is equally applicable to all triangles whatsoever, and is therefore general. But if there is a disruption of the continuity of the visible conditions of the proposition proposed, though the truth which it expresses may be general, it will be necessary to establish the corresponding cases by a distinct demonstration: thus, it is asserted that different parallelograms upon the same base and between the same parallels are equal to one another, and the visible conditions will present three distinct cases for consideration; first, when the sides opposite to the common base have all their points in common: secondly, when the sides opposite to the common base have one point only in common: and thirdly, when the sides opposite to the common base have no points in common: and inasmuch as the terms of the demonstration will be found to involve the notice or consideration of the relative position of the sides opposite to the common base with respect to each other, the precise form of the demonstration, which is applicable to one of these cases, will not be applicable, without alteration, to the other: but if the demonstration of this or any other proposition was conducted through the medium of general symbols, the conclusion obtained, as well as the reasoning by which it was obtained, would be found to be equally general with the symbolical language in which it was expressed.

The separate cases of Geometry are comprehended under the same formula in Algebra.

It is this necessity of considering all the separate cases of the same proposition or problem in Geometry, and the impossibility of comprehending the extreme or limiting, simultaneously with the ordinary, values of the magnitudes which they involve, which commonly renders the reasonings and processes of Geometry more operose and less effective than those which are conducted by the general symbols of Algebra: and it is the permanence of the equivalence of the forms and conclusions expressed by general symbols, for all values of those symbols, whether limiting or otherwise, which are once shewn to be equivalent, when they are general in their form, even though they may be specific in their value (Art. 631), which enables us to include, under one general formula or conclusion, propositions or cases of propositions which must be separately and distinctly considered in Geometry.

The discussion of a general formula, or expression, origi-

nating in the solution of a problem or otherwise, consists in evolving the separate cases which it comprehends, corresponding to the limiting or other values of its symbols, and assigning to each of them their correct interpretation: it requires us to pass in review, the several propositions which, in the geometrical or synthetical exposition of the problem, would require a separate and successive consideration. The geometrical problems which we have considered in this Chapter, present examples of such discussions and analyses, and they will be found generally to constitute the deductive processes which are required in the applications of Algebra to Geometry and Natural Philosophy: it is not easy, therefore, to overrate their importance, or to impress too strongly upon the mind of a student, the necessity of acquiring such clear and accurate notions of the principles of interpretation as may guide him securely in the research and establishment of his conclusions.

CHAPTER XX.

ON THE SOLUTION OF EQUATIONS OF HIGHER ORDERS THAN THE SECOND, WHICH ARE RESOLVIBLE INTO SIMPLE OR QUADRATIC FACTORS.

Classification of equations.

667. EQUATIONS, as we have already seen (Art. 377), are classified according to the highest power of the unknown symbol which they involve, when cleared from fractional and radical expressions: and if we further suppose the significant terms to be transferred to one side, the general forms of simple, quadratic, cubic and biquadratic equations would be as follows.

$$(1)\quad x - p = 0,$$

$$(2)\quad x^2 - px + q = 0,$$

$$(3)\quad x^3 - px^2 + qx - r = 0,$$

$$(4)\quad x^4 - px^3 + qx^2 - rx + s = 0:$$

and similarly for equations of the fifth, sixth and higher orders.

Any equation may be solved which is resolvible into simple or quadratic factors.

668. Any equation may be solved, by processes which have already been considered, if it be resolvible into simple or quadratic factors: for those values of the unknown symbol (x), which severally reduce those factors, and therefore their product, to *zero, and no others*, will satisfy the conditions of the equation: and it is merely necessary to solve the several equations which arise from successively equating those factors to zero, in order to discover all the roots of the equation*.

It is not our intention, in the present Chapter, to attempt to prove the necessary existence of such simple or quadratic factors in all equations reduced to the ordinary form, or the possibility of discovering them, whenever they exist: but we

* Thus if $u = PQR$, and if $u = 0$, then $P = 0$, or $Q = 0$, or $R = 0$: but u cannot become zero, if all its factors P, Q, R retain values different from zero: thus if $u = x^2 - 7x + 12 = (x-3)(x-4)$: then $u = 0$, when $x - 3 = 0$, or $x - 4 = 0$, or when $x = 3$, or $x = 4$: but there is no other value of x which reduces u to zero. Again, let $u = x^3 - 13x^2 + 47x - 35 = (x-1)(x-5)(x-7)$: then $u = 0$ when $x - 1 = 0$, or $x - 5 = 0$, or $x - 7 = 0$: or when $x = 1$, or $x = 5$, or $x = 7$: but every value of x different from these, will make u the product of three significant factors and, therefore, not $= 0$.

shall merely notice the existence of large classes of equations of the superior orders, which admit of such resolution into factors, and consequently of complete solution.

669. To solve the equation The cube roots of 1.

$$x^3 - 1 = 0.$$

Since $x^3 - 1 = (x-1)(x^2+x+1)$, for all values of x (Art. 69, Ex. 5), it follows that

$$x^3 - 1 = 0,$$

when $x - 1 = 0$, or when $x^2 + x + 1 = 0$.

The first equation, $x - 1 = 0$, gives $x = 1$, which is the arithmetical root (Art. 645,) of the primitive equation.

The second equation, or $x^2 + x + 1 = 0$, gives

$$x = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2},$$

or two *unarithmetical* and *imaginary* roots, which also satisfy the primitive equation*.

Inasmuch as the equation $x^3 - 1 = 0$ gives $x^3 = 1$, and therefore $x = \sqrt[3]{1}$, it follows that there are *three* cube roots of 1, which are 1, $\dfrac{-1+\sqrt{3}\sqrt{-1}}{2}$ and $\dfrac{-1-\sqrt{3}\sqrt{-1}}{2}$.

* For $\left(\dfrac{-1\pm\sqrt{3}\sqrt{-1}}{2}\right)^3 = \dfrac{-1 \pm 3\sqrt{3}\sqrt{-1} + 9 \mp 3\sqrt{3}\sqrt{-1}}{8} = \dfrac{8}{8} = 1.$ Their properties.

The square of one of these imaginary roots is equal to the other: for

$$\left(\frac{-1+\sqrt{3}\sqrt{-1}}{2}\right)^2 = \frac{1-2\sqrt{3}\sqrt{-1}-3}{4} = \frac{-1-\sqrt{3}\sqrt{-1}}{2},$$

and $$\left(\frac{-1-\sqrt{3}\sqrt{-1}}{2}\right)^2 = \frac{1+2\sqrt{3}\sqrt{-1}-3}{4} = \frac{-1+\sqrt{3}\sqrt{-1}}{2}.$$

If, therefore, α represents one of these cube roots, α^2 will represent the other, and the three cube roots of 1 may be represented by 1, α, α^2: or they may be represented by $\alpha, \alpha^2, \alpha^3$, since $\alpha^3 = 1$.

Again, since $\alpha \times \alpha^2 = \alpha^3 = 1$, it follows that $\alpha^2 = \dfrac{1}{\alpha}$, and $\alpha = \dfrac{1}{\alpha^2}$: and consequently the three cube roots of 1 may be represented by 1, α and $\dfrac{1}{\alpha}$, where α is one of the imaginary cube roots of 1.

Every integral power of a cube root of 1 is a cube root of 1.

For every integral power of 1 is 1: and since all numbers are expressible by the formulæ $3n$, $3n+1$, $3n+2$ (Art. 526, Note), where n is a whole number, it follows that

$$\alpha^m = \alpha^{3n}, \text{ or } \alpha^{3n+1}, \text{ or } \alpha^{3n+2},$$

where $\alpha^{3n} = 1$; $\alpha^{3n+1} = \alpha^{3n} \times \alpha = 1 \times \alpha = \alpha$;

and $\alpha^{3n+2} = \alpha^{3n} \times \alpha^2 = 1 \times \alpha^2 = \alpha^2$.

The properties of the cube and higher roots of 1 are connected with many important theories, and will be more particularly considered in a subsequent Chapter, (XXIII).

The cube roots of -1.

670. To solve the equation

$$x^3 + 1 = 0.$$

Since $x^3 + 1 = (x+1)(x^2 - x + 1)$, for all values of x, it follows that

$$x^3 + 1 = 0,$$

when $x + 1 = 0$, or when $x^2 - x + 1 = 0$.

The first equation, $x + 1 = 0$, gives $x = -1$.

The second equation, $x^2 - x + 1 = 0$, gives

$$x = \frac{1 \pm \sqrt{3}\sqrt{-1}}{2}.$$

Inasmuch as the equation, $x^3 + 1 = 0$, gives $x^3 = -1$, and therefore $x = \sqrt[3]{-1}$, it follows that the three cube roots of -1, are

$$-1, \quad \frac{1 + \sqrt{3}\sqrt{-1}}{2} \text{ and } \frac{1 - \sqrt{3}\sqrt{-1}}{2} *.$$

The sixth roots of 1.

671. The roots of the equation

$$x^6 - 1 = 0,$$

are included amongst those of

$$x^3 - 1 = 0 \text{ and } x^3 + 1 = 0.$$

For $x^6 - 1 = (x^3 - 1)(x^3 + 1)$ for all values of x, and the roots of $x^6 - 1 = 0$ (Art. 668,) are therefore those of

$$x^3 - 1 = 0 \text{ and } x^3 + 1 = 0.$$

The same expression $x^6 - 1$ may be likewise resolved into the three quadratic factors

$$x^2 - 1, \; x^2 + x + 1 \text{ and } x^2 - x + 1:$$

and the roots of the equations

$$x^2 - 1 = 0, \; x^2 + x + 1 = 0 \text{ and } x^2 - x + 1 = 0,$$

are also the roots of the equation $x^6 - 1 = 0$.

* If a be one of these imaginary roots, the three roots may be expressed by -1, a and a^2, or by -1, a and $\frac{1}{a}$, or by a, a^2 and a^3: for $a^3 = -1$.

672. Biquadratic equations, which present themselves under the form Solution of biquadratic equations wanting the second and fourth terms of the general form in Art. 667.

$$x^4 + qx^2 + s = 0 \qquad (1),$$

are immediately resolvible into the quadratic factors

$$x^2 + \frac{q}{2} + \sqrt{\left(\frac{q^2}{4} - s\right)} \text{ and } x^2 + \frac{q}{2} - \sqrt{\left(\frac{q^2}{4} - s\right)},$$

and consequently admit of solution by the processes given for quadratic equations: for if we make $x^2 = u$, the equation (1) becomes

$$u^2 + qu + s = 0,$$

whose factors are (Art. 662.)

$$u + \frac{q}{2} + \sqrt{\left(\frac{q^2}{4} - s\right)} \text{ and } u + \frac{q}{2} - \sqrt{\left(\frac{q^2}{4} - s\right)},$$

which become the factors above mentioned, when u is replaced by x^2.

The four roots of the proposed biquadratic equation are, therefore,

$$x = \sqrt{\left\{\frac{-q}{2} + \sqrt{\left(\frac{q^2}{4} - s\right)}\right\}},$$

$$x = -\sqrt{\left\{\frac{-q}{2} + \sqrt{\left(\frac{q^2}{4} - s\right)}\right\}},$$

$$x = \sqrt{\left\{\frac{-q}{2} - \sqrt{\left(\frac{q^2}{4} - s\right)}\right\}},$$

$$x = -\sqrt{\left\{\frac{-q}{2} - \sqrt{\left(\frac{q^2}{4} - s\right)}\right\}}.$$

673. The roots of the equation considered in the last Article, may be exhibited under other and equivalent forms, which are sometimes susceptible of more complete arithmetical reduction. The same roots exhibited under a different form.

For if we divide the original equation

$$x^4 + qx^2 + s = 0 \qquad (1)$$

by x^2, we get

$$x^2 + q + \frac{s}{x^2} = 0,$$

and therefore

$$x^2 + \frac{s}{x^2} = -q.$$

But $\left(x + \frac{\sqrt{s}}{x}\right)^2 = x^2 + 2\sqrt{s} + \frac{s}{x^2} = 2\sqrt{s} + x^2 + \frac{s}{x^2}$

$$= 2\sqrt{s} - q;$$

and therefore

$$x+\frac{\sqrt{s}}{x}=\pm\sqrt{(2\sqrt{s}-q)} \qquad (2).$$

If we multiply both sides of this equation by x, we get

$$x^2+\sqrt{s}=\pm\sqrt{(2\sqrt{s}-q)}\,x,$$

and therefore

$$x^2\mp\sqrt{(2\sqrt{s}-q)}\,x=-\sqrt{s}.$$

Solving this pair of quadratic equations in the ordinary manner, we get

$$\left\{x\mp\sqrt{\left(\frac{\sqrt{s}}{2}-\frac{q}{4}\right)}\right\}^2=\frac{\sqrt{s}}{2}-\frac{q}{4}-\sqrt{s}=-\frac{\sqrt{s}}{2}-\frac{q}{4},$$

and therefore

$$x=\pm\sqrt{\left(\frac{\sqrt{s}}{2}-\frac{q}{4}\right)}\pm\sqrt{\left(-\frac{\sqrt{s}}{2}-\frac{q}{4}\right)}*,$$

where the different combinations of the signs $+$ and -1 will furnish the four different roots of the primitive equation (1).

* If we compare the equivalent expressions for x which are obtained in Arts. 672 and 673, we find

$$\sqrt{\left\{-\frac{q}{2}\pm\sqrt{\left(\frac{q^2}{4}-s\right)}\right\}}=\sqrt{\left(\frac{\sqrt{s}}{2}-\frac{q}{4}\right)}\pm\sqrt{\left(-\frac{\sqrt{s}}{2}-\frac{q}{4}\right)},$$

and, consequently,

$$\sqrt{\left(\frac{\sqrt{s}}{2}-\frac{q}{4}\right)}\pm\sqrt{\left(-\frac{\sqrt{s}}{2}-\frac{q}{4}\right)}$$

may be considered as expressing the square roots of $-\frac{q}{2}\pm\sqrt{\left(\frac{q^2}{4}-s\right)}$, a result which may be easily verified: and if we replace $-\frac{q}{2}$ by a, $\frac{q^2}{4}-s$ by b, and therefore s by a^2-b, we shall find

$$\sqrt{(a\pm\sqrt{b}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}\pm\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}:$$

which assumes the form $\sqrt{\alpha}\pm\sqrt{\beta}$ in those cases in which a^2-b is a complete square.

Thus (1) $\sqrt{(5+2\sqrt{6})}=\sqrt{2}+\sqrt{3}$.

(2) $\sqrt{(17-12\sqrt{2})}=3-2\sqrt{2}$.

(3) $\sqrt{(3-4\sqrt{-1})}=2-\sqrt{-1}$.

(4) $\sqrt{2\sqrt{-1}}=1+\sqrt{-1}$.

(5) $\sqrt{\left(\frac{3}{2}+\sqrt{2}\right)}=1+\frac{1}{\sqrt{2}}$.

(6) $\sqrt{(-7+4\sqrt{3})}=2\sqrt{-1}-\sqrt{3}\sqrt{-1}$.

674. Thus, if the equation be Examples.

$$x^4 + 7x^2 + 4 = 0,$$

we get

$$x = \pm \sqrt{\left(-\frac{7}{2} \pm \frac{\sqrt{33}}{2}\right)} = \pm \sqrt{\left(-\frac{7}{4} + \frac{2}{2}\right)} \pm \sqrt{\left(-\frac{7}{4} - \frac{2}{2}\right)}$$

$$= \pm \frac{(\sqrt{-3} \pm \sqrt{-11})}{2}.$$

If the equation be

$$x^4 + 1 = 0,$$

where $q = 0$ and $s = 1$, we get

$$x = \pm \sqrt{\pm \sqrt{-1}} = \pm \frac{(1 \pm \sqrt{-1})}{\sqrt{2}}.$$

If the equation be

$$x^4 - 20x^2 + 64 = 0,$$

we get

$$x = \pm \sqrt{(5 + 4)} \pm \sqrt{(5 - 4)}$$

$$= \pm 3 \pm 1 = 4 \text{ or } -4,\ 2 \text{ or } -2.$$

675. If we extract the square root of the first member of a biquadratic equation arranged as in Art. 667, (where all its significant terms are transposed to one side,) and if we arrive at a remainder, which, with its sign changed, is either a complete square or independent of the unknown symbol, the biquadratic equation may, in all cases, be resolved into two quadratic factors, and its roots determined by the solution of two quadratic equations.

Cases of biquadratic equations which are resolvible into factors by the process of extracting the square root of their first member as arranged in Art. 667.

For if we express by X the first member of the equation, and by u the terms which are obtained in the root before the process of extracting its square root terminates, then $X - u^2$ is the remainder: and if

$$X - u^2 = -v^2,$$

we have

$$X = u^2 - v^2 = (u + v)(u - v),$$

and this is $= 0$, when $u + v = 0$, or when $u - v = 0$ (Art. 668): the roots of the equations $u + v = 0$ and $u - v = 0$ are therefore the roots of the equation $X = 0$.

If v^2 involve x and be not a complete square, then $u+v$, though a factor of X*, does not exhibit x under a rational form: but if v^2 be independent of x, then $u+v$ and $u-v$ will exhibit x under rational forms, whether v^2 be a complete square or not.

Examples. 676. The following are examples.

(1) Let $x^4+\frac{3x^3}{2}-24x-256=0$:

$$\overline{x^4}+\frac{3x^3}{2}-24x-256 \;\Big\{ x^2+\frac{3x}{4}$$

$$2x^2+\frac{3x}{4}\Big\}\quad +\frac{3x^3}{2}+\frac{9x^2}{16}$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$-\frac{9x^2}{16}-24x-256.$$

* The condition, upon which the success of this process of resolution depends, may be easily discovered by extracting the square root of

$$x^4+px^3+qx^2+rx+s,$$

(which is the general form of the first member of a biquadratic equation), as follows:

$$\overline{x^4}+px^3+qx^2+rx+s \;\Big\{ x^2+\frac{px}{2}-\tfrac{1}{2}\left(\frac{p^2}{4}-q\right)$$

$$2x^2+\frac{px}{2}\Big\}\quad +px^3+\frac{p^2x^2}{4}$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$2x^2+px-\tfrac{1}{2}\left(\frac{p^2}{4}-q\right)\Big\}\; -\left(\frac{p^2}{4}-q\right)x^2+rx+s$$

$$-\left(\frac{p^2}{4}-q\right)x^2-\frac{p}{2}\left(\frac{p^2}{4}-q\right)x+\tfrac{1}{4}\left(\frac{p^2}{4}-q\right)^2$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$+\left(\frac{p^3}{8}-\frac{pq}{2}+r\right)x-\tfrac{1}{4}\left(\frac{p^2}{4}-q\right)^2+s$$

The last remainder but one, with its sign changed, or $\left(\frac{p^2}{4}-q\right)x^2-rx-s$, will be a complete square, if $\left(\frac{p^2}{4}-q\right)s=-\frac{r^2}{4}$.

The last remainder will be independent of x, if $\frac{p^3}{8}-\frac{pq}{2}+r=0$.

This last condition will always be satisfied, if $p=0$ and $r=0$, as in the Examples in Art. 674.

The preceding method will succeed in detecting u and v, whenever they are of the form x^2+ax and $bx+c$: or of the form x^2+ax+b and c: but not necessarily when they are of the form x^2+ax+d and $bx+c$, inasmuch as it will generally require the solution of a cubic equation to determine the value of d.

And $\frac{9x^2}{16} + 24x + 256$ is a complete square, since

$$\frac{9}{16} \times 256 = 144 = \tfrac{1}{4} \times (24)^2 \text{*}:$$

and its root is $\frac{3x}{4} + 16$.

The quadratic equations, into which the primitive equation is resolved, are

$$u + v = x^2 + \frac{3x}{4} + \frac{3x}{4} + 16 = x^2 + \frac{3x}{2} + 16 = 0,$$

$$\text{and } u - v = x^2 + \frac{3x}{4} - \frac{3x}{4} - 16 = x^2 - 16 = 0:$$

and the four roots of the biquadratic are

$$-\frac{3 \pm \sqrt{-247}}{4}, \ 4 \text{ and } -4\text{*}.$$

(2) Let the equation be

$$x^4 - 2x^3 + 3x^2 - 2x - 3 = 0.$$

$$\begin{array}{rl}
 & \overline{x^4} - 2x^3 + 3x^2 - 2x - 3 \ (x^2 - x + 1 \\
2x^2 - x) & -2x^3 + x^2 \\
\hline
2x^2 - 2x + 1) & +2x^2 - 2x - 3 \\
 & 2x^2 - 2x + 1 \\
\hline
 & -4 \\
\hline
\end{array}$$

* The following are examples of the same class.

$$(1) \quad x^4 + \frac{13x^3}{3} - 39x - 81.$$

The factors are $x^2 + \frac{13x}{3} + 9$ and $x^2 - 9$: and the roots are 3, -3, and

$$-\frac{13 \pm \sqrt{-155}}{6}.$$

$$(2) \quad x^4 - \frac{2x^3}{3} - \frac{80x^2}{9} + 6x - 1 = 0.$$

The factors are $x^2 + \frac{8x}{3} - 1$ and $x - \frac{10x}{3} + 1$, and the roots are

$$-3, \ \tfrac{1}{3}, \ 3 \text{ and } -\tfrac{1}{3}.$$

We thus get $u^2 = (x^2 - x + 1)^2$ and $v^2 = 4$, and therefore

$$u + v = x^2 - x + 3 = 0,$$

$$u - v = x^2 - x - 1 = 0.$$

The four roots are $\dfrac{1 \pm \sqrt{-11}}{2}$ and $\dfrac{1 \pm \sqrt{5}}{2}$.

(3) Let the equation be

$$x^4 - 2ax^3 + (a^2 - 2b^2)x^2 + 2ab^2x - a^2b^2 = 0.$$

$$\begin{array}{rl} & \overline{x^4} - 2ax^3 + (a^2 - 2b^2)x^2 + 2ab^2x - a^2b^2 \;\big(x^2 - ax - b^2 \\ 2x^2 - ax) & -2ax^3 + a^2x^2 \\ \hline 2x^2 - 2ax - b^2) & -2b^2x^2 + 2ab^2x - a^2b^2 \\ & -2b^2x^2 + 2ab^2x + b^4 \\ \hline & -a^2b^2 - b^4 \end{array}$$

Therefore $u^2 = (x^2 - ax + b^2)^2$ and $v^2 = a^2b^2 + b^4$, and the roots of the component factors

$$x^2 - ax - b^2 + b\sqrt{(a^2 + b^2)} \text{ and } x^2 - ax - b^2 - b\sqrt{(a^2 + b^2)},$$

when respectively equated to zero, are expressed by the formula

$$\frac{a}{2} \pm \sqrt{\left\{\frac{a^2}{4} + b^2 \mp b\sqrt{(a^2 + b^2)}\right\}}\text{*}.$$

* This equation results from the solution of the following problem: "to determine a right-angled triangle, the sum of whose sides containing the right angle is a, and the perpendicular from the right angle upon the hypothenuse is b." If x be taken to represent one of the sides, and therefore $a - x$ the other, the well-known properties of the right-angled triangle will readily furnish the equation

$$\frac{x(a - r)}{b} = \sqrt{(a^2 - 2ax + 2x^2)},$$

$$\text{or } x^2 - ax + b\sqrt{(a^2 - 2ax + 2x^2)} = 0 \qquad (1).$$

This equation is one factor of the *rationalized* equation

$$\{x^2 - ax + b\sqrt{(a^2 - 2ax + 2x^2)}\}\{x^2 - ax - b\sqrt{(a^2 - 2ax + 2x^2)}\}$$
$$= x^4 - 2ax^3 + (a^2 - 2b^2)x^2 + 2ab^2x - a^2b^2 = 0.$$

The two roots $\dfrac{a}{2} \pm \sqrt{\left\{\dfrac{a^2}{4} + b^2 - b\sqrt{(a^2 + b^2)}\right\}}$ belong to the first of these equations, and express the two sides of the triangle which are required to be determined: the other two roots

$$\frac{a}{2} \pm \sqrt{\left\{\frac{a^2}{4} + b^2 + b\sqrt{(a^2 + b^2)}\right\}}$$

belong to the second equation, and are totally foreign to the problem proposed.

This problem is given in the Arithmetica Universalis of Newton (Prob. VII. Cap. II. Sect. IV.) a juvenile work, but which is every where pregnant with marks of the extraordinary genius of its author: the whole section on the Reduction of Geometrical Questions to Equations will richly repay the most careful perusal of the student.

(4) Let the equation be

$$x^2 - 7x + \sqrt{(x^2 - 7x + 18)} = 24.$$

If we transpose 24 and multiply together the two factors

$$x^2 - 7x - 24 + \sqrt{(x^2 - 7x + 18)}$$
$$\text{and } x^2 - 7x - 24 - \sqrt{(x^2 - 7x + 18)},$$

in which the radical expression $\sqrt{(x^2 - 7x + 18)}$ has different signs, their product

$$(x^2 - 7x - 24)^2 - (x^2 - 7x + 18),$$

is rational and becomes *zero* when either of its factors is *zero*.

$$(x^2 - 7x - 24)^2 - x^2 + 7x - 18 \ (x^2 - 7x - 24 - \tfrac{1}{2}$$
$$2(x^2 - 7x - 24) - \tfrac{1}{2}) \ \underline{- x^2 + 7x + 24 + \tfrac{1}{4}}$$
$$- \frac{169}{4}$$

Therefore $(x^2 - 7x - 24 - \frac{1}{2})^2 = \dfrac{169}{4}$,

$$\text{and } x^2 - 7x - 24 - \tfrac{1}{2} = \pm \frac{13}{2}.$$

If we solve this pair of equations, we get the respective pairs of values 9 and -2, and $\dfrac{7 \pm \sqrt{173}}{2}$.

The two roots 9 and -2 belong to the proposed equation: the two others $\dfrac{7 \pm \sqrt{173}}{2}$ belong to the factor

$$x^2 - 7x - 24 - \sqrt{(x^2 - 7x + 18)} = 0,$$

which is introduced for the purpose of rationalizing the proposed equation, and are altogether foreign to it: the first pair of roots may be considered as the *proper* roots of the equation, and the second pair as *roots of solution* merely *.

* Equations, such as that in the text, involving square roots of the unknown symbol or of expressions involving it, may be generally rationalized, by transposing all their terms to one side, and by multiplying together the several factors which arise from changing the signs of the quadratic surds: but it must be kept in mind, that there are generally, though not always, roots of the *rationalized*, which are not roots of the *proposed* equation, and that there are many equations, involving such radical expressions, which admit of no solution whatsoever.

Where the square and simple power only of an expression involving the unknown symbol occur.

677. When equations appear in, or are easily reducible to, a form where the unknown symbol presents itself exclusively in

If the proposed equation had been $x-\sqrt{2x}=4$, the rationalizing factors would be

$$x-4-\sqrt{2x} \text{ and } x-4+\sqrt{2x},$$

and the resulting equation $x^2-10x+16=0$, whose roots are 8 and 2: but it is the first root only which belongs to the proposed equation $x-\sqrt{2x}=4$; the other root 2 belongs to the factor

$$x-4+\sqrt{2x}=0,$$

which is introduced in the process of rationalization, and is foreign therefore to the original equation.

The *adventitious* factors required in the rationalization of the equation

$$3\sqrt{(112-8x)}-19-\sqrt{(3x+7)}=0 \qquad (1),$$

are

$$3\sqrt{(112-8x)}-19+\sqrt{(3x+7)}=0 \qquad (2),$$
$$-3\sqrt{(112-8x)}-19-\sqrt{(3x+7)}=0 \qquad (3),$$
$$-3\sqrt{(112-8x)}-19+\sqrt{(3x+7)}=0 \qquad (4):$$

we first form the two factors, which are the products of (1) and (2) and of (3) and (4) respectively, and which are

$$1362-75x-114\sqrt{112-8x}=0 \qquad (5),$$
$$1362-75x+114\sqrt{112-8x}=0 \qquad (6),$$

and the final rationalized equation is their product or

$$x^2-\frac{11148}{625}x+\frac{44388}{625}=0 \qquad (7),$$

whose roots are 6 and $\frac{7398}{625}$.

The first of these roots is the *proper* root, and corresponds to the primitive equation (1), or to the rationalizing factor (5): the second is a *root of solution*, and corresponds to the factor (6), or to the subordinate factor (4): it may be further observed, that the subordinate factors (2) and (3) are satisfied by neither of these roots, and that they are equations *which admit of no solution whatsoever.*

In a similar manner, the equation

$$\sqrt{(2x+7)}+\sqrt{(3x-18)}=\sqrt{(7x+1)}, \qquad (1),$$

when rationalized, becomes

$$x^2-\frac{27x}{5}-\frac{162}{5}=0, \qquad (2),$$

of whose roots 9 and $-\frac{18}{5}$, the first is the proper root of the proposed equation (1), and the second a root of solution only: also the proper root of the equation

$$7\sqrt{\left(\frac{3x}{2}-5\right)}-\sqrt{\left(\frac{x}{5}+45\right)}-\frac{7}{4}\sqrt{(10x+56)}=0,$$

is 20, and its root of solution $\frac{14568980}{2874649}$: and in both these cases there are irrational factors of the rationalized equation which no value of x can satisfy.

It will be found that 4 and $-\frac{44}{3}$ are proper roots of the equation

$$x+2\sqrt{(x^2+x+5)}-14=0:$$

but there are no values of x which will satisfy its rationalizing factor

$$x-2\sqrt{x^2+x+5)}-14=0.$$

expressions which possess the relation of the square and simple power only, we may replace this expression by a new symbol, and proceed to solve the equation with respect to it.

Thus, if the equation be

$$6x - x^2 + 3\sqrt{(x^2 - 6x + 16)} = 12 \qquad (1),$$

we make $\sqrt{(x^2 - 6x + 16)} = u$, or $x^2 - 6x + 16 = u^2$, and therefore $6x - x^2 = 16 - u^2$: we thus get the quadratic equation

$$16 - u^2 + 3u = 12 \qquad (2),$$

where the values of u are 4 and -1: but inasmuch as the form of the proposed equation (1) excludes a negative value of u, we confine our attention to the arithmetical root of equation (2)*: we thus get

$$u = \sqrt{x^2 - 6x + 16} = 4,$$

$$\text{or } x^2 - 6x + 16 = 16,$$

$$\text{and } x = 0 \text{ or } 6,$$

which are the *proper* roots of the equation.

The equation

$$\{(x+3)^2 + (x+3)\}^2 - 7(x+3)^2 = 7x + 711$$

is easily reducible to the form

$$\{(x+3)^2 + (x+3)\}^2 - 7\{(x+3)^2 + (x+3)\} = 690,$$

which becomes, if we make $u = (x+3)^2 + (x+3)$

$$u^2 - 7u = 690.$$

The values of u are 30 and -23: those of x are 2, -9, $\dfrac{-17 \pm \sqrt{-91}}{2}$, which are the four roots of the proposed biquadratic equation †.

In the equation

$$\frac{x}{x^2 + x + 5} + \frac{5}{\sqrt{(x^2 + x + 5)}} = \frac{116}{25x}$$

* If we take the negative value of u or -1, the resulting roots $3 \pm \sqrt{-6}$, correspond to the equation

$$6x - x^2 - 3\sqrt{x^2 - 6x + 16} = 12,$$

and are merely *roots of solution* of the original equation.

† The biquadratic equation, if reduced to the ordinary form, is

$$x^4 + 14x^3 + 66x^2 - 119x - 630 = 0,$$

which may be solved by the extraction of its square root.

we multiply both sides by x, and thus get

$$\frac{x^2}{x^2+x+5} + \frac{5x}{\sqrt{(x^2+x+5)}} = \frac{116}{25}:$$

if we make $\frac{x}{\sqrt{(x^2+x+5)}} = u$, the roots of the resulting equation

$$u^2 + 5u = \frac{116}{25},$$

are $\frac{4}{5}$ and $-\frac{29}{5}$, of which the second must be rejected.

The roots of the equation

$$u = \frac{x}{\sqrt{(x^2+x+5)}} = \frac{4}{5},$$

are 4 and $-\frac{20}{9}$, of which the second must be rejected, being merely a root of solution: there is, therefore, only *one* proper root of the proposed equation.

Let $x^{\frac{6}{5}} + 6x^{\frac{3}{5}} = 891$.

Make $u = x^{\frac{3}{5}}$, and we get

$$u^2 + 6u = 891,$$

$$u = x^{\frac{3}{5}} = 27, \text{ or } -33,$$

$$\therefore\ x^{\frac{1}{5}} = (1)^{\frac{1}{3}} \times 3, \text{ or } (-1)^{\frac{1}{3}}\sqrt[3]{33}\,*,$$

$$x = (1)^{\frac{5}{3}} \times 3^5, \text{ or } (-1)^{\frac{5}{3}}\,.\,(33)^{\frac{5}{3}}$$

$$= (1)^{\frac{1}{3}} \times 243, \text{ or } (-1)^{\frac{1}{3}}\,.\,\sqrt[3]{39135393}\,\dagger.$$

The number of proper solutions of an equation not affected by a denominator of the index of the unknown symbol.

678. The number of proper solutions of an equation is not affected by a common denominator of the index of the unknown symbol, or its powers: for if the highest power of this symbol was $x^{\frac{p}{n}}$, the process of solution would give us the values of $x^{\frac{1}{n}}$, and the number of those values would not be increased by the transition from the values of $x^{\frac{1}{n}}$ to those of x: thus the equation

$$x^6 + 6x^3 = 891$$

* There are three cube roots of 1, which are 1, $\frac{-1 \pm \sqrt{3}\sqrt{-1}}{2}$ (Art. 669), and three cube roots of -1, which are -1, $\frac{-1 \pm \sqrt{3}\sqrt{-1}}{2}$ (Art. 670): there are, therefore, six values of $x^{\frac{1}{5}}$ in this equation.

† For $(1)^{\frac{5}{3}} = (1)^{\frac{1}{3}}$ and $(-1)^{\frac{5}{3}} = (-1)^{\frac{1}{3}}$.

would have the same number of proper solutions with the equation

$$x^{\frac{6}{5}} + 6x^{\frac{3}{5}} = 891,$$

and the solutions themselves only differ in the roots of one equation being the fifth powers of those of the other.

If, however, we should suppose the equation

$$x^{\frac{6}{5}} + 6x^{\frac{3}{5}} = 891$$

to admit equally all the forms which are proper to the different symbolical values of $x^{\frac{1}{5}}$ (which will be found hereafter to be 5 in number,) we should have 5 equations and 30 roots, of which 6 only are the proper roots of the proposed equation, the rest being roots of solution.

It will appear, likewise, that the reduction of the indices of the unknown symbol, when one or more of them are fractional, to a common denominator, will convert roots of solution into proper roots, unless the change from one form to the other is accompanied by a limitation of the roots to be extracted, in consequence of such a change, to their arithmetical values only: thus the *proper* root of the equation

$$x + \sqrt{x} = 6 \qquad (1)$$

is 4, and 9 is the root of solution: but 4 and 9 are equally *proper* roots of the equation

$$x^{\frac{2}{2}} + x^{\frac{1}{2}} = 6 \qquad (2),$$

unless we are equally restricted to the arithmetical value of $x^{\frac{1}{2}}$ in the two equations (1) and (2).

In a similar manner, there are only *three proper* roots of the equation

$$x^3 - 6x^{\frac{3}{2}} - 16 = 0 \qquad (1),$$

but there are *six proper* roots of the equation

$$x^{\frac{6}{2}} - 6x^{\frac{3}{2}} - 16 = 0 \qquad (2),$$

which include the roots of the former equation (1).

CHAPTER XXI.

THE BINOMIAL THEOREM AND ITS APPLICATIONS.

The binomial theorem when the index is a whole number.

679. In Chap. VIII. (Art. 486...), we have proved, when the index n is a whole number, that

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{1.2} + n(n-1)(n-2)\frac{x^3}{1.2.3} + \ldots :$$

and it will be seen, from an examination of this series, and of the law of its formation, that the powers of x and their divisors are independent of n, and that the coefficients of

$$x, \frac{x^2}{1.2}, \frac{x^3}{1.2.3}, \ldots\ldots, \frac{x^r}{1.2\ldots r},$$

are n, $n(n-1)$, $n(n-1)(n-2)$, ..., $n(n-1)\ldots(n-r+1)$,

being, for the $(1+r)^{\text{th}}$ term, the continued product of the descending series of natural numbers from n to $n-r+1$.

The same series is equivalent when the index is perfectly general in value as well as in form.

680. This series for $(1+x)^n$ is perfectly general in its form, though n is specific in its value, and it will continue therefore, by "the principle of the permanence of equivalent forms" (Art. 631) to be equivalent to $(1+x)^n$, when n is general in value as well as in form: and it will consequently admit, in virtue of this equivalence, of being immediately translated into the whole series of propositions respecting indices and their interpretation, which are given in Chapter XVI *.

The product of the series for $(1+x)^n$ and $(1+x)^{n'}$ is the series for $(1+x)^{n+n'}$.

681. Thus "the general principle of indices" (Art. 635) shews that

$$(1+x)^n (1+x)^{n'} = (1+x)^{n+n'}$$

for all values of n and n', and consequently the product of the series for $(1+x)^n$ or

$$1 + nx + n(n-1)\frac{x^2}{1.2} + n(n-1)(n-2)\frac{x^3}{1.2.3} + \&c.,$$

* See Appendix.

and of the series for $(1+x)^{n'}$, or

$$1 + n'x + n'(n'-1)\frac{x^2}{1.2} + n'(n'-1)(n'-2)\frac{x^3}{1.2.3} + \&c.$$

will be equivalent to the series for $(1+x)^{n+n'}$, or

$$1 + (n+n')x + (n+n')(n+n'-1)\frac{x^2}{1.2}$$
$$+ (n+n')(n+n'-1)(n+n'-2)\frac{x^3}{1.2.3} + \&c,$$

under the same circumstances.

682. Again, the series for $\frac{1}{(1+x)^n}$, or $(1+x)^{-n}$ (Art. 640,) will be found by replacing n by $-n$ in the series for $(1+x)^n$: we thus get The series for $(1+x)^{-n}$.

$$(1+x)^{-n} = 1 - nx + n(n+1)\frac{x^2}{1.2} - n(n+1)(n+2)\frac{x^3}{1.2.3} + \&c:$$

for the product $-n(-n-1)$ may be replaced by $n(n+1)$: the product $-n(-n-1)(-n-2)$ by $-n(n+1)(n+2)$, and similarly for the subsequent terms.

683. The series for $(1-x)^n$ is deducible, in virtue of the same principle (Art. 631), from that of $(1+x)^n$, by changing the signs of those terms which involve the odd powers of x: we thus get, as in Art. 491, The series for $(1-x)^n$,

$$(1-x)^n = 1 - nx + n(n-1)\frac{x^2}{1.2} - n(n-1)(n-2)\frac{x^3}{1.2.3} + \ldots:$$

and in a similar manner, the series for $(1-x)^{-n}$ will become and for $(1-x)^{-n}$.

$$1 + nx + n(n+1)\frac{x^2}{1.2} + n(n+1)(n+2)\frac{x^3}{1.2.3} + \ldots,$$

where all the signs and terms are positive*.

684. Since $A + Au = A(1+u)$, and therefore Series for $(A+Au)^n$.

$$(A+Au)^n = A^n(1+u)^n,$$

* Thus $(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$$

$$(1-x)^{-3} = 1 + 3x + \frac{3.4x^2}{1.2} + \frac{3.4.5x^3}{1.2.3} + \ldots\ldots$$

it follows that

$$(A+Au)^n = A^n\left\{1+nu+n(n-1)\frac{u^2}{1.2}+n(n-1)(n-2)\frac{u^3}{1.2.3}+\&c.\right\}$$

Examples. (1) Thus, if $A + Au = a^2 + x^2 = a^2\left(1+\frac{x^2}{a^2}\right)$, we have

$$A = a^2 \text{ and } u = \frac{x^2}{a^2},$$

$$\text{and } A^n = a^{2n},\ u^2 = \frac{x^4}{a^4},\ u^3 = \frac{x^6}{a^6},\ \ldots\ldots;$$

$$\text{therefore } (a^2+x^2)^n = a^{2n}\left\{1+n\,.\,\frac{x^2}{a^2}+n(n-1)\,.\,\frac{x^4}{1\,.\,2a^4}\right.$$

$$\left.+\,n(n-1)(n-2)\,.\,\frac{x^6}{1\,.\,2\,.\,3a^6}+\ldots\right\}$$

$$= a^{2n}+na^{2n-2}x^2+n(n-1)\frac{a^{2n-4}x^4}{1\,.\,2}+n(n-1)(n-2)\frac{a^{2n-6}x^6}{1\,.\,2\,.\,3}+\ldots$$

multiplying a^{2n} into every term of the series.

(2) If $A + Au = a^2 - ax = a^2\left(1-\frac{x}{a}\right)$, we get

$$A = a^2,\ u = -\frac{x}{a},\ u^2 = \frac{x^2}{a^2},\ u^3 = -\frac{x^3}{a^3},$$

and therefore

$$(a^2-ax)^n = a^{2n}\left\{1-n\frac{x}{a}+n(n-1)\frac{x^2}{1.2a^2}-n(n-1)(n-2)\frac{x^3}{1.2.3a^3}+\ldots\right.$$

$$\left.= a^{2n}-na^{2n-1}x+n(n-1)a^{2n-2}\frac{x^2}{1.2}-n(n-1)(n-2)a^{2n-3}\frac{x^3}{1.2.3}+\ldots\right\}$$

Series for $(1+x)^{\frac{1}{2}}$. 685. Since $(1+x)^{\frac{1}{2}}$ means the square root of $1 + x$ (Art. 636,) it follows that the corresponding series {replacing n by $\frac{1}{2}$ in the series for $(1+x)^n$},

$$1+\tfrac{1}{2}\,.\,x+\tfrac{1}{2}(\tfrac{1}{2}-1)\frac{x^2}{1\,.\,2}+\tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)\frac{x^3}{1\,.\,2\,.\,3}+\ldots$$

expresses the square root of $1 + x$.

If in this series we replace the factors

$$\frac{1}{2}-1 \text{ by } -\frac{1}{2},\ \frac{1}{2}-2 \text{ by } -\frac{3}{2},\ \frac{1}{2}-3 \text{ by } -\frac{5}{2},$$

we get

$$(1+x)^{\frac{1}{2}} = 1+\tfrac{1}{2}x-\frac{1\times1}{2^2}\,.\,\frac{x^2}{1.2}+\frac{1\times1\times3}{2^3}\,.\,\frac{x^3}{1.2.3}-\frac{1\times1\times3\times5}{2^4}\,.\,\frac{x^4}{1.2.3.4}$$

$$= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\ldots$$

If the series for $(a^2+x^2)^{\frac{1}{2}} = a\left(1+\dfrac{x^2}{a^2}\right)^{\frac{1}{2}}$ be required, we get Series for $(a^2+x^2)^{\frac{1}{2}}$.

$$(a^2+x^2)^{\frac{1}{2}} = a\left(1+\frac{x^2}{2a^2}-\frac{x^4}{8a^4}+\frac{x^6}{16a^6}-\frac{5x^8}{128a^8}+\dots\right)$$

$$= a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\dots$$

In a similar manner, we shall find Series for $(a^2-x^2)^{\frac{1}{2}}$.

$$(a^2-x^2)^{\frac{1}{2}} = a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-\dots$$

These results are identical with those given in Art. 650, Ex. 1 and 2, and the student will not fail to observe the readiness with which the terms of the series are formed and the law of their formation ascertained, compared with the ordinary process of extracting the square root.

686. Since $(1+x)^{\frac{1}{3}}$ means the cube root of $1+x$ (Art. 637), it follows that the corresponding series {replacing n by $\frac{1}{3}$ in the series for $(1+x)^n$}, The series for $(1+x)^{\frac{1}{3}}$.

$$1+\frac{1}{3}x+\frac{1}{3}\left(\frac{1}{3}-1\right)\frac{x^2}{1.2}+\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\frac{x^3}{1.2.3}+\dots$$

expresses the same cube root of $1+x$.

This series becomes, by replacing

$$\frac{1}{3}-1 \text{ by } -\frac{2}{3},\ \frac{1}{3}-2 \text{ by } -\frac{5}{3},\ \frac{1}{3}-3 \text{ by } -\frac{8}{3}\dots,$$

$$1+\frac{1}{3}x-\frac{1\times 2}{3^2}\cdot\frac{x^2}{1.2}+\frac{1.2.5}{3^3}\cdot\frac{x^3}{1.2.3}-\frac{1.2.5.8}{3^4}\cdot\frac{x^4}{1.2.3.4}+\dots$$

$$= 1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}-\frac{10x^4}{243}+\dots$$

687. More generally, if we replace n by $\dfrac{p}{q}$ in the series for $(A+Au)^n$ (Art. 684), and replace the factors $\dfrac{p}{q}-1$, $\dfrac{p}{q}-2$, $\dfrac{p}{q}-3,\dots$ of the numerators of the successive coefficients by $\dfrac{p-q}{q}$, $\dfrac{p-2q}{q}$, $\dfrac{p-3q}{q}$, ... we shall find The series for $(A+Au)^{\frac{p}{q}}$.

$$(A+Au)^{\frac{p}{q}} = A^{\frac{p}{q}}\left\{1+\frac{p}{q}\cdot u+\frac{p(p-q)}{q^2}\cdot\frac{u^2}{1.2}\right.$$

$$\left.+\frac{p(p-q)(p-2q)}{q^3}\cdot\frac{u^3}{1.2.3}+\dots\right\}.$$

Examples. Thus, in the expansion or developement of $(a^2+ax)^{\frac{3}{10}}$, where

$$A=a^2,\ u=\frac{x}{a},\ p=3,\ p-q=-7,\ p-2q=-17,\ p-3q=-27,\ldots$$

we get

$$(a^2+ax)^{\frac{3}{10}}=a^{\frac{3}{5}}\left(1+\frac{3}{10}\cdot\frac{x}{a}-\frac{3\,.\,7}{10^2}\cdot\frac{x^2}{1\,.\,2a^2}+\frac{3\,.\,7\,.\,17}{10^3}\cdot\frac{x^3}{1\,.\,2\,.\,3a^3}\right.$$
$$\left.-\frac{3\,.\,7\,.\,17\,.\,27}{10^4}\cdot\frac{x^4}{1\,.\,2\,.\,3\,.\,4\,.\,a^4}+\ldots\right)$$
$$=a^{\frac{3}{5}}+.3\times\frac{x}{a^{\frac{2}{5}}}-.105\times\frac{x^2}{a^{\frac{7}{5}}}+.0595\times\frac{x^3}{a^{\frac{12}{5}}}-.0401625\times\frac{x^4}{a^{\frac{17}{5}}}+\ldots$$

by multiplying $a^{\frac{3}{5}}$ into every term and reducing the several numerical coefficients.

The developement of $(a-x)^{-\frac{1}{10}}$, where

$$A=a,\ u=-\frac{x}{a},\ p=-1,\ p-q=-11,\ p-2q=-21,\ldots$$

gives

$$(a-x)^{-\frac{1}{10}}=a^{-\frac{1}{10}}\left(1+\frac{1}{10}\cdot\frac{x}{a}+\frac{1\,.\,11}{10^2}\cdot\frac{x^2}{1\,.\,2a^2}+\frac{1\,.\,11\,.\,21}{10^3}\cdot\frac{x^3}{1\,.\,2\,.\,3a^3}\right)$$
$$=\frac{1}{a^{\frac{1}{10}}}+.1\times\frac{x}{a^{\frac{11}{10}}}+.055\times\frac{x^2}{a^{\frac{21}{10}}}+.0385\times\frac{x^3}{a^{\frac{31}{10}}}+{}^*\ldots$$

Exponential and terminal coefficients. 688. The coefficients of the simple powers of x in the series for $(1+x)^n$ consist, as we have already seen (Art. 679), of two parts, one dependent upon the index or exponent of the power of the binomial, and the other upon the position of the term in the series: and it will be convenient to mark this fundamental distinction in the origin of the factors of the entire coefficient, by calling one of them the *exponential*, and the other the *terminal* coefficient of the series: thus, the *exponential* and *terminal* coefficients of the $(1+r)^{\text{th}}$ term of the series for $(1+x)^n$ are

* Other examples of the same kind are,

(1) $$\frac{1}{\sqrt{(1+x)}}=(1+x)^{-\frac{1}{2}}=1-\tfrac{1}{2}x+\frac{1\,.\,3}{2^2}\cdot\frac{x^2}{1\,.\,2}-\frac{1\,.\,3\,.\,5}{2^3}\cdot\frac{x^3}{1\,.\,2\,.\,3}+\&c.$$
$$=1-\tfrac{1}{2}x+\frac{3x^2}{8}-\frac{5x^3}{16}+\frac{35x^4}{128}-\frac{63x^5}{256}+\cdots$$

(2) $$(a^6+a^3x^2)^{\frac{2}{5}}=a^2\left\{1+\frac{2}{5}\cdot\frac{x^2}{a^2}-\frac{2\,.\,3}{5^2}\cdot\frac{x^4}{1\,.\,2a^4}+\frac{2\,.\,3\,.\,8}{5^3}\cdot\frac{x^6}{1\,.\,2\,.\,3\,.\,a^6}-\&c.\right\}$$
$$=a^2+.4\times x^2-.12\times\frac{x^4}{a^2}+.064\times\frac{x^6}{a^4}-.0416\times\frac{x^8}{a^6}+\cdots$$

$$n(n-1)\ldots(n-r+1) \text{ and } \frac{1}{1\times2\times3\ldots r} \text{ respectively,}$$

the entire coefficient being formed by their product, which is

$$\frac{n(n-1)\ldots(n-r+1)}{1\times2\times3\times\ldots r}.$$

689. The factors of the numerator of the *exponential* coefficient, when the index is $\frac{p}{q}$, form a decreasing arithmetical series, of which the first term is p, and the common difference is q (Art. 687); whilst its denominator is that power of q, whose index is the number of such factors: thus in the developement of $(1+x)^{\frac{3}{10}}$, this arithmetical series is $3, -7, -17, -27$: and the *exponential* coefficient of the 5th term is

Rule for the formation and reduction of the coefficient, when the index is fractional.

$$\frac{3\times-7\times-17\times-27}{10^4} = -\frac{3\times7\times17\times27}{10^4},$$

the number of factors being one less than the denomination of the term: the corresponding *terminal* coefficient is $\frac{1}{1\times2\times3\times4}$, and the *complete* coefficient required is $-\frac{3\times7\times17\times27}{1\times2\times3\times4\times10^4}$, or .0401625.

Again, in the developement of $(1+x)^{-\frac{3}{10}}$, the arithmetical series is $-3, -13, -23, -33$, and the complete coefficient of the 5th term is $\frac{3\times13\times23\times33}{1\times2\times3\times4\times10^4}$, or .1233625.

These examples will be sufficient to shew, that when the arithmetical series of the factors of the numerator of the *exponential* coefficient is once formed, the law of formation of the *complete* coefficients becomes immediately manifest*.

* Thus the 7th term of $(a^7+a^6x)^{\frac{1}{7}}$, where $A = a^7$ and $u = \frac{x}{a}$, is

$$-\frac{1\times6\times13\times20\times27\times34}{7^6\times1\times2\times3\times4\times5\times6}A^{\frac{1}{7}}u^6 = .01604\times\frac{x^6}{a^5}:$$

the 4th term of $(\sqrt{x}-\sqrt{a})^{-\frac{1}{4}}$, where $A = x^{\frac{1}{2}}$, $u = -\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$, is

$$\frac{1\times5\times9}{4^3\times1\times2\times3}\cdot A^{-\frac{1}{4}}u^3 = .1171875\times\frac{a^{\frac{3}{2}}}{x^{\frac{13}{8}}}$$

The limits of the inverse ratio of any two consecutive coefficients of the series for $(1+x)^n$.

690. The inverse ratio of any two consecutive complete coefficients of the series for $(1+x)^n$, such as the r^{th} and the $(1+r)^{\text{th}}$, will be expressed by $\frac{n-r+1}{r} = \frac{n+1}{r} - 1$ *: and inasmuch as the limits of the values of r are 1 and *infinity* (∞), (or n, when n is a positive whole number), it follows that the limits of the values of this ratio are n and -1, and that, when the series does not terminate, it perpetually approximates to the latter.

The limits of the inverse ratio of any two consecutive terms of the series for $(1+x)^n$.

691. In a similar manner it will appear that the inverse ratio of any two consecutive terms of the series for $(1+x)^n$ is expressed by $\frac{n-r+1}{r} \cdot x = \left(\frac{n+1}{r} - 1\right) x$, the extreme limits of which are nx and $-x$; and the series for $(1+x)^n$ will therefore, sooner or later, when it does not terminate, approximate in form and character to the series

The series for $(1+x)^n$ approximates sooner or later to a geometric series.

$$T - Tx + Tx^2 - Tx^3 + \ldots$$

which *converges* or *diverges* according as x is greater or less than 1. (Art. 432, and Note.)

Points of convergency or divergency in the binomial series.

692. This character of convergency or divergency of the eries for $(1+x)^n$ will be dependent upon the value of the inverse ratio $\left(\frac{n-r+1}{r}\right) x$ of two of its consecutive terms: if this ratio be less than 1 (without regard to its sign, whether + or −), it is *convergent:* if greater than 1, it is *divergent:* and the point of change from divergency to convergency, or the contrary, will take place in the $(1+r)^{\text{th}}$ term, where r is that whole number, which is next greater than $\frac{(n+1)x}{1+x}$, or than $\frac{(n+1)x}{x-1}$ †, according as $\frac{n+1}{r} - 1$ is positive or negative: thus,

* For the complete coefficient of the r^{th} term is $\frac{n(n-1)\ldots(n-r+2)}{1.2.3\ldots(r-1)}$, and of the $(1+r)^{\text{th}}$ term is $\frac{n(n-1)\ldots(n-r+2)(n-r+1)}{1.2.3\ldots(r-1)r}$, and the quotient of the second divided by the first is $\frac{n-r+1}{r}$.

† If $\left(\frac{n+1}{r} - 1\right) x = 1$, we get $r = \frac{(n+1)x}{x+1}$; but if $\left(\frac{n+1}{r} - 1\right) x = -1$, we get $r = \frac{(n+1)x}{x-1}$: in cases, where $\frac{n+1}{r} - 1$ passes from positive to negative,

if $n = \frac{3}{2}$ and $x = \frac{9}{10}$, then r is the next whole number, which is greater than $\frac{45}{38}$, and therefore the series becomes convergent from its second term: if $n = -3$ and $x = \frac{11}{12}$, we find

$$r = \frac{(-3+1)\frac{11}{12}}{\frac{11}{12} - 1} = 22,$$

and the convergency begins from the 23rd term: if $n = \frac{13}{4}$ and $x = \frac{4}{3}$, we have a point of convergency when $r = 3$, and, subsequently, a point of divergency, when $r = 17$: if $n = -\frac{1}{2}$ and $x = \frac{10}{9}$, we find $r = 5$, or the divergency begins with the 6th term of the series.

The series for $(1+x)^n$ and $(1-x)^n$ are convergent or divergent according as $1+x$ and $1-x$ are arithmetical or not, both in their arrangement and value.

693. It appears, therefore, that the series for $(1+x)^n$ and $(1-x)^n$ are or become, convergent, whenever x is less than 1, or whenever the binomials, whose powers are developed, are arithmetical, both in their arrangement and value: but that they cease to be so whenever x is greater than 1, or whenever they cease to be arithmetical either in their value or their arrangement: for in this case $1-x$ is negative, and therefore unarithmetical in its value: whilst under the same circumstances, the binomial $1+x$, though arithmetical in its value, is not so in its arrangement, inasmuch as the greater term succeeds the less in an inverse and interminable operation (Art. 386)*: but it

and x is greater than 1, we may obtain positive values of r from both these formulæ, which shews that a point of convergency is in such a case followed by a point of divergency.

* It is in the inverse operations of Arithmetic and Arithmetical Algebra, such as Division and Evolution, that we meet with interminable results, and in which the arrangement of the terms of the expressions, which are the subjects of the operations, in the order of their magnitude, is absolutely necessary to enable us to approximate to, when we cannot accurately obtain, the true result which is sought for: such an arrangement, which is emphatically called *arithmetical*, is not absolutely necessary, though generally convenient, in operations, whether direct or inverse, which lead to a terminable result.

should be observed, that in the latter case, the terms of the resulting series, either are, or become, alternately positive and negative, confirming the remark which has elsewhere been made (Arts. 517 and 650), that the expression in which it originates, though arithmetical in its value, is not so in the character of the operation to which it is subjected.

694. If the series which arises from the developement of $(1+x)^n$ be arithmetical, or convergent, we shall be enabled not only to approximate to its true value or sum (s) by the aggregation of its terms (provided we include the term from which the convergency begins), but likewise to assign the limits of the excess or defect of the aggregate thus formed from the true sum which is required.

Thus if σ be the aggregate or sum of r terms of the series, and T, with its proper sign, its $(1+r)^{\text{th}}$ term, then the true sum s will differ from σ by a quantity less than

$$\frac{\frac{n+1}{r}xT}{\left\{1+\left(1-\frac{n+1}{r}\right)x\right\}(1+x)}\ ^*.$$

Thus $(1+x)^2 = 1+2x+x^2$ and $(x+1)^2 = x^2+2x+1$, and the results

$$1+2x+x^2 \text{ and } x^2+2x+1$$

are identical in their arithmetical value, though not in their arrangement, whether x be less or greater than 1: but $(1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots$ and

$$(x+1)^{-2} = \frac{1}{x^2}\left\{1-\frac{2}{x}+\frac{3}{x^2}-\frac{4}{x^3}+\ldots\right\};$$

and if x be less than 1, it is the first, and if x be greater than 1, it is the second, of these series which is convergent: or, in other words, the convergency or divergency of the resulting series depends upon the arithmetical or unarithmetical arrangement of the terms of the binomial.

* For if $\rho = 1-\frac{n+1}{r}$, it will follow, since $1-\frac{n+1}{r}$ increases as r increases, that the sum of the geometrical series $T-T\rho x+T\rho^2x^2-T\rho^3x^3+\ldots$, which is $\frac{T}{1+\rho x}$ (Art. 432), will be less than the sum σ' of the corresponding terms

$$T-T\left(1-\frac{n+1}{r}\right)x+T\left(1-\frac{n+1}{r}\right)\left(1-\frac{n+1}{1+r}\right)x^2-\ldots$$

of the series for $(1+x)^n$; but the sum of the geometrical series

$$T-Tx+Tx^2-Tx^3+\ldots,$$

which is $\frac{T}{1+x}$, will be greater than σ' since ρ is necessarily less than 1, when r is greater than n: it follows, therefore, inasmuch as σ' is intermediate in value between

As an Example, suppose it was required to determine within what limits of error the square root of 5 would be given by the aggregation of 5 terms of the series for $2(1+\frac{1}{4})^{\frac{1}{2}}$, or Example.

$$2+\tfrac{1}{4}-\frac{1}{4\times 4^2}+\frac{1}{2\times 4\times 4^3}-\frac{5}{4^3\times 4^4}+\frac{7}{2\times 4^3\times 4^5},$$

we should find

$$\sigma = 2.236026 \text{ nearly,}$$

$$T = .0000534 \text{ nearly,}$$

$$\frac{\frac{(n+1)}{r}Tx}{\left\{1+\left(1-\frac{n+1}{r}\right)x\right\}(1+x)} = .0000321 \text{ nearly,}$$

$$\frac{T}{1+x}=\frac{4T}{5}=.0000427 \text{ nearly,}$$

$$\sigma+\frac{T}{1+x}=2.2360687 \text{ nearly,}$$

and therefore s differs from 2.2360687 by a quantity less than .0000321.

$\frac{T}{1+\rho x}$ and $\frac{T}{1+x}$, that $\sigma+\frac{T}{1+x}$ will be less and $\sigma+\frac{T}{1+\rho x}$ will be greater, than s or $\sigma+\sigma'$ by a quantity less than

$$\frac{T}{1+\rho x}-\frac{T}{1+x}=\frac{(1-\rho)Tx}{(1+\rho x)(1+x)}=\frac{\frac{n+1}{r}Tx}{\left\{1+\left(1-\frac{n+1}{r}\right)x\right\}(1+x)}.$$

CHAPTER XXIII.

ON THE USE OF THE SQUARE AND HIGHER ROOTS OF 1 AS SIGNS OF AFFECTION.

The signs of affection hitherto required and recognized, are the four biquadratic roots of 1.

695. THE signs of affection, which we have hitherto used, are $+$, $-$, $\sqrt{-1}$ and $-\sqrt{-1}$; the two first of which $+$ and $-$ may be conveniently replaced by $+1$ and -1, if we consider them, like the two last, as factors of the symbols which they affect*. Thus $+a = +1 \times a$ and $-a = -1 \times a$, in the same manner that $a\sqrt{-1} = \sqrt{-1} \times a$ and $-a\sqrt{-1} = -\sqrt{-1} \times a$ (Art. 652). Of these signs, the two first $+1$ and -1, or 1 and -1, are the two square roots of 1: whilst the two last, $\sqrt{-1}$ and $-\sqrt{-1}$ are the two square roots of -1: the entire series of them, $+1$, -1, $+\sqrt{-1}$ and $-\sqrt{-1}$, may be easily shewn to be the four biquadratic roots of 1 †.

Their use in generalizing the ordinary operations of Arithmetical Algebra.

696. The preceding signs, or their equivalents, are sufficient, as we have shewn in the preceding Chapters, to express every symbolical consequence which arises from extending the rules for Addition, Subtraction, Multiplication, Division, and the Extraction of the square root, which are proved in Arithmetical Algebra, *to all values whatsoever of the symbols to which they are applied, in conformity with the general principle of the permanence of equivalent forms* (Art. 631): no further signs of affection would be required, if the processes of Arithmetical and Symbolical Algebra, were confined to the several operations above enumerated.

Corresponding use of the higher orders of the roots of 1 in the higher processes of Evolution.

697. But the processes of Evolution in Arithmetic and Arithmetical Algebra, are not confined to the extraction of square and biquadratic roots, and it will be found necessary to introduce new signs, in order to give the same extension to the rules for the extraction of cubic and higher roots, and which, like those we have already considered, are capable, as we shall proceed to shew, of being correctly expressed or symbolized by the multiple symbolical values of the cubic and higher roots of 1.

* See Appendix.

† For $x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0$: the roots of the equation $x^2 - 1 = 0$, are 1 and -1: those of $x^2 + 1 = 0$, are $\sqrt{-1}$ and $-\sqrt{-1}$.

698. Thus $a^3 = 1 \times a^3$, and therefore $\sqrt[3]{a^3} = \sqrt[3]{1} \times a$: and, inasmuch as we have shewn (Art. 669), that the three cube roots of 1 are In the case of the cube roots of 1.

$$1, \quad \frac{-1+\sqrt{3}\sqrt{-1}}{2}, \quad \frac{-1-\sqrt{3}\sqrt{-1}}{2},$$

it will follow that there are also three cube roots of a^3, which are

$$a, \quad \frac{-1+\sqrt{3}\sqrt{-1}}{2}a, \quad \frac{-1-\sqrt{3}\sqrt{-1}}{2}a^*,$$

the first of which alone is arithmetical.

699. In a similar manner, we have $-a^3 = -1 \times a^3$, and therefore $\sqrt[3]{-a^3} = \sqrt[3]{-1} \times a$: and inasmuch as we have shewn that the three cube roots of -1 (Art. 670), are Of the cube roots of -1.

$$-1, \quad \frac{1+\sqrt{3}\sqrt{-1}}{2}, \quad \frac{1-\sqrt{3}\sqrt{-1}}{2},$$

it will follow that there are also three cube roots of $-a^3$, which are

$$-a, \quad \frac{1+\sqrt{3}\sqrt{-1}}{2}a, \quad \frac{1-\sqrt{3}\sqrt{-1}}{2}a,$$

none of which are arithmetical.

700. It may be easily shewn that the three cube roots of 1, and the three cube roots of -1, form the six senary roots of 1: for if $x = \sqrt[6]{1}$, we have $x^6 = 1$, and therefore The cube roots of 1 and -1 are the senary roots of 1.

$$x^6 - 1 = 0 \qquad (1):$$

and inasmuch as

$$x^6 - 1 = (x^3 - 1)(x^3 + 1),$$

* For $\left(\frac{-1+\sqrt{3}\sqrt{-1}}{2}a\right)^3 = \left(\frac{-1+\sqrt{3}\sqrt{-1}}{2}\right)^3 a^3 = a^3,$

for $\left(\frac{-1+\sqrt{3}\sqrt{-1}}{2}\right)^3 = 1$ (Art. 669, Note):

and also

$$\left(\frac{-1-\sqrt{3}\sqrt{-1}}{2}a\right)^3 = \left(\frac{-1-\sqrt{3}\sqrt{-1}}{2}\right)^3 a^3 = a^3,$$

for $\left(\frac{-1-\sqrt{3}\sqrt{-1}}{2}\right)^3 = 1.$

it follows that the equation (1) is equally satisfied by the three roots of $x^3 - 1 = 0$, which are the three cube roots of 1, and by the three roots of $x^3 + 1 = 0$, which are the three cube roots of -1. (Art. 671).

Use of the n^{th} roots of 1 and -1.

701. More generally, since

$$a^n = 1 \times a^n, \text{ and } -a^n = (-1)\, a^n,$$

it follows that

$$\sqrt[n]{a^n} = \sqrt[n]{1} \times a \text{ and } \sqrt[n]{-a^n} = \sqrt[n]{-1} \times a,$$

and the n^{th} roots of a^n and $-a^n$ (which will be shewn hereafter to be severally n in number), will be correctly expressed or symbolized by multiplying a into the several symbolical values of $\sqrt[n]{1}$ in one case, and of $\sqrt[n]{-1}$ in the other*.

It thus appears that the several symbolical values of the different orders of the roots of 1 and -1 will enable us to give to the processes of evolution generally the full extension which the conditions of Symbolical Algebra require: and we shall proceed, in the next Chapter, to investigate some of the more important properties of those roots, with a view to the establishment of the principles upon which their use and interpretation shall be founded.

* The n^{th} roots of 1 and of -1 form the $2n^{th}$ roots of 1: for if $x = \sqrt[2n]{1}$, we have

$$x^{2n} - 1 = 0 \qquad (1),$$

and therefore

$$x^{2n} - 1 = (x^n - 1)(x^n - 1);$$

and it is obvious that the equation (1) is equally satisfied by the n roots of $x^n - 1 = 0$ and by the n roots of $x^n + 1 = 0$.

CHAPTER XXIV.

ON THE SYMBOLICAL PROPERTIES OF DIFFERENT ORDERS OF THE ROOTS OF 1.

702. IF we represent a root of 1, which is of the n^{th} order, by x, its n^{th} power or x^n will be equal to 1: its values will therefore be the roots of the equation $x^n - 1 = 0$, whatever they may be: the discovery or determination therefore of the roots of this equation is equivalent to the discovery or determination of the corresponding roots of 1. The n roots of 1 are the roots of the equation $x^n - 1 = 0$.

703. One of these roots is always 1, which is the only arithmetical root: for it may be easily shewn that There is only one arithmetical root of the equation $x^n - 1 = 0$.

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots x + 1)\ *$$

which becomes *zero* when x is 1: but there is no arithmetical value of x, which, under any circumstances, can make the second factor of this product $x^{n-1} + x^{n-2} + \dots x + 1$ equal to *zero*.

The discussion of the properties of the roots of this equation, in those special cases in which they can be easily determined and exhibited, will form the best introduction to their general theory.

704. The roots of the equation $x^2 - 1 = 0$, which are the square roots of 1, are 1 and -1, one of which only is arithmetical. The square roots of 1.

If we represent the unarithmetical root or -1 by a, the two roots of the equation will be expressed by a and a^2: for Their properties.

$$a^2 = (-1)^2 = 1.$$

705. The roots of the equation $x^3 - 1 = 0$, which are the cube roots of 1, are The cube roots of 1.

$$1, \quad \frac{-1 + \sqrt{3}\sqrt{-1}}{2}, \quad \frac{-1 - \sqrt{3}\sqrt{-1}}{2}:$$

* For if we divide $x^n - 1$ by $x - 1$, the *complete* quotient is

$$x^{n-1} + x^{n-2} + \dots\dots x + 1;$$

and since all its terms are positive, there is no positive or arithmetical value of x which can make the sum of its terms equal to *zero*.

Their properties.

and if a represent one of the unarithmetical roots, the three roots will be represented by a, a^2, a^3 (Art. 669, Note) *.

The biquadratic roots of 1.

706. The roots of the equation $x^4 - 1 = 0$, which are the biquadratic roots of 1, are 1, -1, $\sqrt{-1}$ and $-\sqrt{-1}$ (Art. 695, Note). The two first are also the square roots of 1: the two last are the square roots of -1, or of that square root of 1, which is not arithmetical: and if a represent $\sqrt{-1}$ or $-\sqrt{-1}$, the four roots will be expressed by a, a^2, a^3 and a^4.

The quinary roots of 1.

707. The roots of the equation $x^5 - 1 = 0$, which are the quinary roots of 1, are

$$1, \quad \frac{\sqrt{5}-1}{4} \pm \sqrt{\left(\frac{-5-\sqrt{5}}{8}\right)}, \quad -\left(\frac{\sqrt{5}+1}{4}\right) \pm \sqrt{\left(\frac{-5+\sqrt{5}}{8}\right)} \dagger,$$

$$\text{or } 1, \quad .309 \pm 951\sqrt{-1}, \quad -.809 \pm 587\sqrt{-1} \text{ nearly.}$$

Their properties.

It appears that the pairs of imaginary roots are severally reducible to the form $a \pm b\sqrt{-1}$, and that in both cases

$$\sqrt{(a^2 + b^2)} = 1 \ddagger:$$

* Or by a, $\frac{1}{a}$ and 1, one of the imaginary or unarithmetical roots being the reciprocal of the other.

† For $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1) = 0$, where

$$x - 1 = 0 \text{ or } x^4 + x^3 + x^2 + x + 1 = 0.$$

In order to solve the equation $x^4 + x^3 + x^2 + x + 1 = 0$, we divide it by x^2, which gives

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0,$$

$$\text{or } x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0;$$

$$\text{or } \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 1, \quad \text{(Art. 677).}$$

$$\text{or } \left(x + \frac{1}{x} + \frac{1}{2}\right)^2 = \frac{5}{4},$$

$$\text{or } x + \frac{1}{x} + \frac{1}{2} = \frac{\pm\sqrt{5}}{2},$$

$$\text{or } x + \frac{1}{x} = \frac{\pm\sqrt{5} - 1}{2}:$$

the solution of this quadratic equation gives the values of x in the text.

‡ For if $a = .309$ and $b = .951$, we find

$$\sqrt{(a^2 + b^2)} = \sqrt{.999882} = .9999 \ldots\ldots = 1 \text{ nearly}:$$

or if $a = \frac{\sqrt{5}-1}{4}$ and $b = \sqrt{\left(\frac{5+\sqrt{5}}{8}\right)}$, we get $a^2 + b^2 = \frac{3-\sqrt{5}}{8} + \frac{5+\sqrt{5}}{8} = 1$,

this will be shewn, in a subsequent Chapter, to be a property common to the imaginary roots of 1, whatever be their order*.

If we represent one of the quinary roots of 1, which is different from 1, by a, their whole series will be expressed by

$$a,\ a^2,\ a^3,\ a^4,\ a^5;$$

and if the same series be continued indefinitely, the same succession of values will be reproduced *periodically*†.

This is a general property of those imaginary roots of 1, which are denominated by *prime* numbers, and will be demonstrated generally in the following proposition.

708. If there exists a root‡ a of the equation $x^n - 1$, which is different from 1, when n is a prime number, then its n roots will be expressed by the terms of the series General property of the n^{th} roots of 1 when n is a prime number.

$$a,\ a^2,\ a^3,\ \ldots a^n \qquad (1).$$

In the first place, no power of a, whose index is less than n, can be equal to 1.

and similarly, if $a = -.809$ and $b = .587$, we get

$$\sqrt{(a^2+b^2)} = \sqrt{.999639} = .9999 = 1 \text{ nearly:}$$

or if $a = \dfrac{\sqrt{5}+1}{4}$ and $b = \sqrt{\left(\dfrac{5-\sqrt{5}}{8}\right)}$, we get $a^2 + b^2 = \dfrac{3+\sqrt{5}}{8} + \dfrac{5-\sqrt{5}}{8} = 1$.

* Thus the pair of the imaginary roots of $x^3 - 1 = 0$ is $-\frac{1}{2} \pm \dfrac{\sqrt{3}\sqrt{-1}}{2}$, where $a = -\frac{1}{2}$ and $b = \dfrac{\sqrt{3}}{2}$: we therefore get $\sqrt{(a^2+b^2)} = \sqrt{(\frac{1}{4}+\frac{3}{4})} = 1$.

† Thus if $a = .309 + .951\sqrt{-1}$ be one of the unarithmetical roots of $x^5 - 1 = 0$, we find

$$a^2 = -.809 + .587\sqrt{-1},$$
$$a^3 = -.809 - .587\sqrt{-1},$$
$$a^4 = .309 - .951\sqrt{-1},$$
$$a^5 = 1,$$

which are also the quinary roots of 1: it will be found likewise, if the series be continued, that

$$a^6 = a^5 \times a = 1 \times a = a,$$
$$a^7 = a^5 \times a^2 = 1 \times a^2 = a^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

or that the same series of values is reproduced, and so on for ever.

‡ It will be demonstrated, in a subsequent Chapter, that there exists in all cases *one*, and therefore $n - 1$, roots of the equation $x^n - 1 = 0$, which are different from each other and from 1: in the absence of such a demonstration, the existence of such roots in the equations $x^3 - 1 = 0$, $x^4 - 1 = 0$ and $x^5 - 1 = 0$, where they have been determined, affords a reasonable presumption that they exist likewise in all other cases.

For, if not, let $a^p = 1$, where p is less than n: then since $a^p - 1 = 0$ and $a^n - 1 = 0$, for the same value of a, it follows that they must have a common factor, which becomes zero for that value: but if n be a prime number, $a - 1$ is the only common factor of $a^p - 1$ and $a^n - 1$, and which can only become zero when $a = 1$ (Art. 703), which is contrary to the hypothesis: it follows, therefore, that a^n is the lowest power of a, which is equal to 1.

In the second place, the successive powers of a, whose indices are less than n, are different from each other.

For, if not, let $a^p = a^q$, where q is less than p, and both of them less than n: dividing both sides by a^q, we get $a^{p-q} = 1$, which is impossible, since the index $p - q$ is less than p, and therefore less than n.

In the third place, the first n terms of the series (1) are roots of the equation $x^p - 1 = 0$.

For $a^n = 1$, since a is assumed to be a root of the equation.

Again, since

$$(a^2)^n = (a^n)^2 = (1)^2 = 1,$$
$$(a^3)^n = (a^n)^3 = (1)^3 = 1,$$
$$(a^p)^n = (a^n)^p = (1)^p = 1,$$
$$(a^n)^n = (1)^n = 1,$$

it follows that a^2, $a^3 \ldots\ldots a^n$, satisfy the requisite conditions of the equation equally with a: and since they are all of them different from each other, and n in number, the equation $x^n - 1 = 0$ can have no other roots.

The indefinite series formed by the successive powers of an imaginary root of $x^n - 1 = 0$ is periodic.

709. If the series

$$a,\ a^2,\ a^3,\ \ldots\ a^n$$

be continued beyond a^n, forming the terms

$$a^{n+1},\ a^{n+2},\ a^{n+3},\ \ldots$$

the same values will recur in the same order.

$$\text{For } a^{n+1} = a^n \times a = 1 \times a = a,$$
$$a^{n+2} = a^n \times a^2 = 1 \times a^2 = a^2,$$
$$a^{n+3} = a^n \times a^3 = 1 \times a^3 = a^3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It appears, therefore, that the series

$$a,\ a^2,\ \ldots\ a^n,\ a^{n+1},\ a^{n+2}, \ldots$$

is *periodic*, the same terms recurring, in the same order, after every n^{th} term.

710. The root (a) of the equation $x^n - 1$, whose successive powers form the *complete period* Base of the period and their number.

$$a, a^2, \ldots a^n$$

may be called its *base:* and it is obvious that there are as many *bases* as there are roots of the equation different from 1.

It will be farther shewn, in the articles which follow, that if n be a *composite* number, no root of the equation $x^n - 1 = 0$ will possess a similar property; but that in all such cases, a *base* of a complete period of n terms may be formed, which is dependent upon those roots of 1 different from 1, which are, severally denominated by the prime factors of n.

711. The roots of the equation The roots of the equation $x^{pq} - 1 = 0$ are the products of the roots of the subordinate equations $x^p - 1 = 0$ and $x^q - 1 = 0$.

$$x^{pq} - 1 = 0 \qquad (1)$$

when p and q are *different* prime numbers, are the several products which can be formed of the roots of the subordinate equations $x^p - 1 = 0$, and $x^q - 1 = 0$.

For, let α and β be two roots of $x^p - 1 = 0$, and $x^q - 1 = 0$ respectively, which are different from 1, and let

$$\alpha, \alpha^2, \ldots \alpha^p \qquad (2),$$

$$\beta, \beta^2, \ldots \beta^q \qquad (3),$$

be their corresponding periods; then the several terms of their product, which are n in number, form all the roots of the equation $x^{pq} - 1 = 0$.

For if $\alpha^r\beta^s$ be any one of those terms, we have

$$(\alpha^r\beta^s)^{pq} = (\alpha^{pq})^r (\beta^{pq})^s = 1^*:$$

it is therefore a root of the equation $x^{pq} - 1 = 0$.

Again, all the terms of this product are different from each other: for, if not, let $\alpha^r\beta^s = \alpha^{r'}\beta^{s'}$, and therefore $\alpha^{r-r'} = \beta^{s'-s}$ †, which is impossible, since 1 is the only root which is common to the equations $x^p - 1 = 0$, and $x^q - 1 = 0$.

The terms of this product, therefore, which are all of them different from each other and pq in number, form all the roots of the equation $x^{pq} - 1 = 0$.

* For $\alpha^{pq} = (\alpha^p)^q = 1$, since $\alpha^p = 1$: and $\beta^{pq} = (\beta^q)^p = 1$, since $\beta^q = 1$.

† For $r - r'$ is less than p, and $s' - s$ is less than q: and if $s' - s$ be negative, we get $\beta^{s'-s} = \dfrac{1}{\beta^{s-s'}} = \dfrac{\beta^q}{\beta^{s-s'}} = \beta^{q-(s-s')}$, which is a root of $x^q - 1 = 0$, and different from 1, since $q - (s - s')$ is less than q.

The product $\alpha\beta$ is the base of a complete period of the roots of $x^{pq}-1=0$.

712. The product $\alpha\beta$ is the *base* of a *complete period* of pq terms of the roots of the equation $x^{pq}-1=0$.

In the first place every term of the period

$$\alpha\beta,\ (\alpha\beta)^2,\ (\alpha\beta)^3 \ldots (\alpha\beta)^{pq}, \ldots$$

is equal to some one term of the product of the subordinate periods corresponding to $x^p-1=0$, and $x^q-1=0$.

For $(\alpha\beta)^r=\alpha^r\beta^r$, and α^r is always a root of $x^p-1=0$, and β^r a root of $x^q-1=0$, whatever be the value of r: and therefore $\alpha^r\beta^r$ is, by the last Article, always a root of $x^{pq}-1=0$.

Again, all the terms of this period are different from each other.

For, if not, let $(\alpha\beta)^r=(\alpha\beta)^s$, where r is greater than s: we thus get $\alpha^{r-s}=\beta^{s-r}=\beta^{tq-(r-s)}$ (Art. 711, note), which is impossible, unless

$$r-s=mp, \text{ and } tq-r-s=nq, \text{ or } mp=n'q, \text{ where } n'=t-n\,;$$

and since p is prime to q, the least values of m and n' in this equation are q and p (Arts. 110 and 116): it follows, therefore, that $r-s$ is not less than pq, and consequently the terms $(\alpha\beta)^r$ and $(\alpha\beta)^s$ which are equal to each other are beyond the limits of the period.

The terms, therefore, of this period, which are pq in number, form all the roots of the equation $x^{pq}-1=0$.

The roots of $x^6-1=0$.

713. Thus the roots of the equation $x^6-1=0$, are found by multiplying together the three roots of $x^3-1=0$ and the two roots of $x^2-1=0$: they are therefore

$$1,\ \frac{-1+\sqrt{3}\sqrt{-1}}{2},\ \frac{-1-\sqrt{3}\sqrt{-1}}{2},\ -1,\ \frac{1-\sqrt{3}\sqrt{-1}}{2},\ \frac{1+\sqrt{3}\sqrt{-1}}{2}.$$

The same series of roots, though not in the same order, form the terms of the period whose base $(\alpha\beta)$ is

$$-1\times\frac{-1+\sqrt{3}\sqrt{-1}}{2} \text{ or } \frac{1-\sqrt{3}\sqrt{-1}}{2}.$$

The roots of $x^{p^2}-1=0$.

714. If p and q be equal to each other, as in the equation

$$x^{p^2}-1=0,$$

the series of roots in the subordinate equations

$$x^p-1=0 \text{ and } x^q-1=0$$

become identical with each other, and no new values or roots

are formed therefore by their multiplication with each other: but under such circumstances, the roots of the equation

$$x^{p^2}-1=0,$$

will be found to be the p^{th} roots of those of the equation

$$x^p-1=0.$$

For if we make $x^p=u$, we get $x^{p^2}=u^p=1$, and therefore $x=\sqrt[p]{u}$, where the values of u are the roots of the equation $x^p-1=0$.

715. Again, if we suppose α to be one of the roots of $x^p-1=0$, which is different from 1, and β one of the p^{th} roots of α, we shall find that the p^2 roots of the equation $x^{p^2}-1=0$ may be represented by the terms of the period whose base is $\alpha\beta$. The period which they form.

For the terms of this period, which are p^2 in number, are all different from each other.

For if not, let $(\alpha\beta)^r=(\alpha\beta)^s$, and therefore $\alpha^{r-s}=\beta^{s-r}$ where r and s are less than p: and since $\alpha^{r-s}=\beta^{p(r-s)}$, it follows that $\beta^{(p+1)(r-s)}=1$: but since $r-s$ cannot exceed $p-1$, it follows that $(p+1)(r-s)$ cannot exceed $(p+1)(p-1)$, or p^2-1, and therefore $\beta^{(p+1)(r-s)}$ cannot be equal to 1: it will follow therefore that the terms of the period $\alpha\beta$, $(\alpha\beta)^2$, ... $(\alpha\beta)^{p^2}$ form all the roots of the equation $x^{p^2}-1=0$.

716. Thus the roots of the equation $x^9-1=0$ are the cube roots of the roots of the equation $x^3-1=0$: they are therefore the cube roots of The roots of $x^9-1=0$.

$$1,\ \frac{-1+\sqrt{3}\sqrt{-1}}{2},\ \frac{-1-\sqrt{3}\sqrt{-1}}{2}.$$

The method of extracting the cube and higher roots of expressions, such as $\dfrac{-1+\sqrt{3}\sqrt{-1}}{2}$ and $\dfrac{-1-\sqrt{3}\sqrt{-1}}{2}$, will be explained in a subsequent Chapter.

717. If p, q, r be three prime factors of n, which are different from each other, and if α, β, γ be severally roots of the equations $x^p-1=0$, $x^q-1=0$, and $x^r-1=0$, which are different from 1, then the period of n terms, whose *base* is $\alpha\beta\gamma$, will form all the roots of the equation $x^n-1=0$. The roots and periods of the equations $x^{pqr}-1=0$, and $x^{p^3}-1=0$.

In a similar manner, if $n=p^3$ and if α be a root of $x^p-1=0$, which is different from 1, and if β is $\sqrt[p]{\alpha}$, and if γ is $\sqrt[p]{\beta}$, we

shall find that the period of p^3 terms, whose base is $\alpha\beta\gamma$, will express all the roots of the equation $x^{p^3} - 1 = 0$.

It is not necessary to give the demonstration of these propositions, nor to shew in what manner they may be extended to the exhibition of the roots, periods, and their bases of the equation $x^n - 1 = 0$, when n is any *composite* number whatsoever.

General conclusion.

It appears, therefore, that the roots of the equation $x^n - 1 = 0$ will form, in all cases, *periods* of n terms, which are the successive powers of a common *base:* and it is this remarkable property, which will be shewn in a subsequent Chapter, to form the principal basis of their interpretation.

The roots of $x^n + 1 = 0$.

718. The roots of the equation $x^n + 1 = 0$, which are the n roots of -1, are included amongst the roots of the equation

$$x^{2n} - 1 = 0\,*.$$

Their periods.

If a be one of the roots of $x^n + 1 = 0$, which is different from -1, the odd powers of the base a of the period of $2n$ terms

$$a,\ a^2,\ a^3,\ \ldots\ a^{2n-1},\ a^{2n}\ldots$$

will express the whole series of its roots.

For $a^n = -1$, $(a^3)^n = (a^n)^3 = (-1)^3 = -1$:

$$(a^5)^n = (a^n)^5 = (-1)^5 = -1,$$

and so on for other terms of the series: it follows therefore that $a, a^3, a^5, \ldots$ are roots of $x^n + 1 = 0$.

Again, all the terms, and therefore the *odd* terms of the period

$$a,\ a^2,\ a^3\ \ldots\ a^{2n}$$

are different from each other: and since they are n in number, they form all the roots of the equation $x^n + 1 = 0$.

The roots of $x^4 + 1 = 0$, and of $x^8 - 1 = 0$.

719. Thus the roots of the equation $x^4 + 1 = 0$ are included amongst those of the equation $x^8 - 1 = 0$: and if $\dfrac{1+\sqrt{-1}}{\sqrt{2}}$ be one of the roots of $x^4 + 1 = 0$, all its roots are expressed by the period

$$\frac{1+\sqrt{-1}}{\sqrt{2}},\ \left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^3,\ \left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^5,\ \left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^7,$$

* For $x^{2n} - 1 = (x^n - 1)(x^n + 1)$.

which are equivalent to

$$\frac{1+\sqrt{-1}}{\sqrt{2}},\ \frac{-1+\sqrt{-1}}{\sqrt{2}},\ \frac{-1-\sqrt{-1}}{\sqrt{2}},\ \frac{1-\sqrt{-1}}{\sqrt{2}}.$$

The series formed by the *even* powers of the same base (Art. 710) $\frac{1+\sqrt{-1}}{\sqrt{2}}$, which are $\sqrt{-1}$, -1, $-\sqrt{-1}$ and 1, are the roots of the equation $x^4-1=0$. (Art. 695).

720. We have shewn (Arts. 708 and 710), that when the index n is a prime number, the same period of n terms may arise from $(n-1)$ different bases, which are the $(n-1)$ first terms of the period, the last or n^{th} term being 1: but it will likewise be found that the terms of all these periods follow a different order: thus, if in the period The order of succession of the terms of a period is altered by a change of its base.

$$a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ 1 \qquad (1),$$

corresponding to $x^7-1=0$, we replace the base a by a^2, we shall get the period

$$a^2,\ a^4,\ a^6,\ a,\ a^3,\ a^5,\ 1 \qquad (2),$$

whose terms are the 2nd, 4th, 6th, 1st, 3rd, 5th and 7th terms of the period from which it is derived: if in the same period (1) we replace a by a^3, we shall get the period

$$a^3,\ a^6,\ a^2,\ a^5,\ a,\ a^4,\ 1 \qquad (3),$$

whose terms are the 3rd, 6th, 2nd, 5th, 1st, 4th and 7th of the same period: and it may be observed generally, that if in a series of repeating periods of n terms, we replace the first by the r^{th} term, then if n be prime to r, we shall reproduce a period involving the same terms, with that from which it is derived, but disposed in a different order, being such as would arise from taking every r^{th} term of the periodic series*.

* For if one term (the p^{th}) of the series of r^{th} terms coincides with a final term a^n or 1 of a (the q^{th}) period, in the periodic series $a, a^2 \dots a^n, a, a^2 \dots a^n \dots$ we must have $pr=qn$ or $\frac{p}{q}=\frac{n}{r}$, and since n is prime to r, the fraction $\frac{n}{r}$ cannot be reduced to lower terms: and since p and q are the least possible numbers, which will answer the required conditions, it will follow that $p=n$ and $q=r$: or in other words, it is the n^{th} term of the new period which coincides with the final term of the r^{th} period of the periodic series.

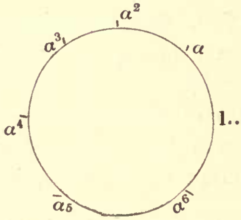
Again, all the terms of the period thus produced are different from each other: for, if possible, let $a^{sr}=a^{tr}$, and therefore $a^{(s-t)r}=1$: but since s and t are both of

Properties of such a period if arranged circularly.

721. If the n terms, therefore, of a period be arranged *circularly*, and if we mark in succession every r^{th} term, (r being prime to n), we shall pass round the circle r times, having marked every term of the period *once*, and *once* only, before we return to the first term: thus, if $n=7$ and $r=2$, and if we arrange the terms of the period

$$a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ 1 \qquad (1),$$

round the circumference of a circle, and mark every second term, beginning from 1 the succession or new period of *marked* terms will be



$$a^2,\ a^4,\ a^6,\ a,\ a^3,\ a^5,\ 1 \qquad (2),$$

where every term of the first period is included *once*, and *once* only, and where 2 circuits are made before we return to 1: again, if we make $r=5$, we shall form the period

$$a^5,\ a^3,\ a,\ a^6,\ a^4,\ a^2,\ 1 \qquad (3),$$

where 5 circuits are made before we return to 1: and similarly in other cases: and it is obvious that the same periods will be found, and in the same order, if we replace a, in every successive term of the period (1), by a^2 in one case, and by a^5 in the other, depressing the indices in every case to the *residual* which arises from dividing them by 7.

Cyclical arrangement of the roots of $\dfrac{x^n-1}{x-1}=0$, where n is a prime number.

722. If we restrict our attention to the *imaginary* roots of $x^n-1=0$, where n is a prime number, or, in other words, to the roots of the equation

$$\frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+\dots.\ x+1=0,$$

we shall find that they are capable of an arrangement, by which the terms of the period which they form, may always recur in the same order, upon the replacement of one base by another; this may be effected by arranging the terms of the period

$$a,\ a^2,\ a^3,\ \dots a^{n-1}$$

them less than n, it follows that $s-t$ is less than n: but we have shewn above, that the least power of a^r, which is equal to 1, is a^{nr}.

It may be observed, that the proposition is true of all powers of a base whose indices are prime to n, whether n be a prime number or not.

in such an order that their indices may be the residuals of the successive powers of a *primitive* root of n (Art. 531), which comprehend every number from 1 to $n-1$ inclusive: thus, if n be 7, of which 3 is a *primitive* root, the residuals of

$$3,\ 3^2,\ 3^3,\ 3^4,\ 3^5,\ 3^6,$$

are

$$3,\ 2,\ 6,\ 4,\ 5,\ 1^*:$$

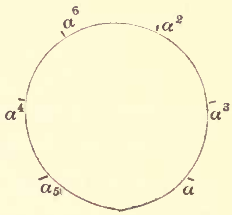
and if the terms of the period

$$a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6 \qquad (1),$$

corresponding to $\dfrac{x^7-1}{x-1}=0$, be distributed in the order of these residuals, as follows,

$$a^3,\ a^2,\ a^6,\ a^4,\ a^5,\ a \qquad (2),$$

and then arranged circularly, it will be found that the succession of its terms will remain unaltered, whatever be the base or term of the period by which the assumed base or a is replaced: thus, if a be replaced by a^3, we get the period



$$a^2,\ a^6,\ a^4,\ a^5,\ a,\ a^3 \qquad (3),$$

whose terms follow the same circu-

* These residuals are successively formed by multiplying the preceding residual by the primitive root, or 3: thus, 3 being the first residual, the second is 2, which is the remainder from dividing 3×3 by 7: the third is 3×2 or 6: the fourth is 4, the remainder from dividing 3×6 by 7: the fifth is 5, the remainder from dividing 3×4 by 7: the sixth is 1, the remainder from dividing 3×5 by 7: they afterwards recur in the same order for ever. If we take the second primitive root of 7, which is 5, we shall get the series of residuals 5, 4, 6, 2, 3, 1, the five first terms of one series being the five first terms of the other series in a reverse order.

If $n = 13$, there are 4 primitive roots corresponding, which give the following periods of residuals:

$$2 \text{ and } 7 \begin{cases} 2,\ 4,\ 8,\ 3,\ 6,\ 12,\ 11,\ 9,\ 5,\ 10,\ 7,\ 1, \\ 7,\ 10,\ 5,\ 9,\ 11,\ 12,\ 6,\ 3,\ 8,\ 4,\ 2,\ 1: \end{cases}$$

$$6 \text{ and } 11 \begin{cases} 6,\ 10,\ 8,\ 9,\ 2,\ 12,\ 7,\ 3,\ 5,\ 4,\ 11,\ 1, \\ 11,\ 4,\ 5,\ 3,\ 7,\ 12,\ 2,\ 9,\ 8,\ 10,\ 6,\ 1: \end{cases}$$

these periods form pairs, in which the first 11 terms follow severally an inverse order: and it may be observed, that the terms of the second pair are the 5th terms of those of the first.

The student will find the theory of such *primitive* roots discussed at considerable length in Art. 531, and those which precede and follow it.

lar arrangement with those of the period (2), each term being one place in advance: if we replace a by a^{3^2} or a^2, we get

$$a^6,\ a^4,\ a^5,\ a,\ a^3,\ a^2 \qquad (4),$$

where each term is two places in advance, when compared with the same period (2): if we replace a by a^{3^3} or a^6, we get the period

$$a^4,\ a^5,\ a,\ a^3,\ a^2,\ a^6 \qquad (5),$$

where each term is three places in advance: and similar results will be observed to follow, whatever be the term in the series by which a is replaced.

Cyclical periods: their great importance.

723. Such periods of the imaginary roots of 1, which correspond to prime values of the index, may be properly called *cyclical* periods, inasmuch as the various derivative periods which result from changes of their bases, will perpetually follow in the same order round the circumference of a circle, where no regard is paid to the initial term: they will be found hereafter to be connected with the most important analytical theories*, and therefore deserve the most careful attention of the student.

* They form the basis of Gauss' well-known researches respecting the geometrical division of the circle, and of Lagrange's theory of the solution of the equation $x^n - 1$ for all values of n. See the *Disquisitiones Arithmeticæ* of Gauss, Sectio 7ma, and the *Résolution des Equations Numériques* of Lagrange, Note xiv.

CHAPTER XXV.

ON THE GENERAL PRINCIPLES OF THE INTERPRETATION OF THE SIGNS OF AFFECTION WHICH ARE SYMBOLIZED BY THE ROOTS OF 1.

724. We have explained in a former Chapter (XXIII.) the introduction and use of the roots of 1, as the *recipients* of the signs of affection, which the application of the general principle of the "permanence of equivalent forms" (Art. 631) renders necessary in algebraical operations, and more particularly in those of Evolution: we have shewn that $1 \times r$ and $-1 \times r$ may be conveniently used as equivalent to $+r$ and $-r$ respectively, where 1 and -1, which are also the square roots of 1, may be considered as the *recipients* of the signs $+$ and $-$: the extraction of the square roots of expressions, such as r^2 and $-r^2$, or of their equivalents $1 \times r^2$ and $-1 \times r^2$, leads to results which are correctly symbolized by $\sqrt{1} \times r$ and $\sqrt{-1} \times r$: the consideration of higher roots conducts us in a similar manner to expressions such as $\sqrt[n]{1} \times r$ and $\sqrt[n]{-1} \times r$, or to their equivalents $(1)^{\frac{1}{n}} \times r$ and $(-1)^{\frac{1}{n}} r$, where the signs $(1)^{\frac{1}{n}}$ and $(-1)^{\frac{1}{n}}$ are used to designate such affections or qualities of their common subject r, as can be shewn to be consistent with their symbolical properties.

The use of the roots of 1 as the recipients of signs of affection in Symbolical Algebra.

725. We have subsequently investigated (Chapter XXIV.) the more important of these properties, with a view to the discovery of the conditions which must be made the basis of their interpretation: we have shewn, when n is a prime number, that $(1)^{\frac{1}{n}}$ has necessarily n symbolical equivalents or values*, provided it has one such value which is different from 1: and we have further demonstrated that these equivalent values or roots may be considered as the successive powers of a common

Recapitulation of their symbolical properties.

* The term *value*, when thus used, has no reference to magnitude, but to symbolical form only, or to the quality of magnitude which its symbolical conditions are competent to express: all the values of $(1)^{\frac{1}{n}} r$ are *equal* in magnitude, but *different* in affection or quality.

base, which is one of them, forming periods of n terms, which recur in the same order for ever.

Research of the conditions of their interpretation when applied to specific magnitudes.

726. If r, therefore, be a specific magnitude to which the sign $(1)^{\frac{1}{n}}$ is attached, or into which it is multiplied, forming the expression $(1)^{\frac{1}{n}} r$; and if a be a *base* of $(1)^{\frac{1}{n}}$ (Art. 710), or any one of its roots which is different from 1, then $(1)^{\frac{1}{n}} r$ may equally express any one term of the period

$$a r, \; a^2 r, \; a^3 r, \ldots a^{n-1}, \; a^n r \text{ or } r \qquad (1),$$

which are all of them symbolically different from each other.

The successive terms of a period possess the same symbolical relation to each other:

727. It will be observed that the successive terms of this period bear the same symbolical relation to each other, the ratios

$$\frac{a^2 r}{a r}, \; \frac{a^3 r}{a^2 r}, \; \frac{a^4 r}{a^3 r}, \ldots \frac{a^n r}{a^{n-1} r},$$

being identical with each other, and with a: and it will follow, therefore, that whatever be the affection or quality which a is capable of symbolizing, when applied to the specific magnitude r, it must equally symbolize the affection which connects $a^2 r$ with $a r$, $a^3 r$ with $a^2 r$, and so on, throughout the period, until we reach its last term $a^n r$, which is identical with r.

and must represent therefore quantities of the same kind and of the same magnitude.

We may conclude, therefore, that the quantities designated by the terms of the period

$$a r, \; a^2 r, \; a^3 r \ldots a^n r$$

must be of the *same kind**, and of the *same magnitude*†; but inasmuch as the successive terms of this period are symbolically different from each other, they are also different in their affections: but if we enter upon a second period, the same terms will re-appear, and in the same order: and so on from period to period, however far the series of them may be extended.

Their geometrical interpretation.

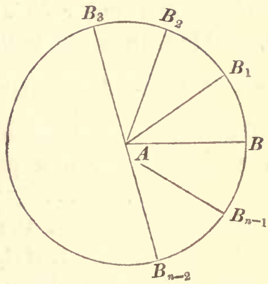
728. The relative position of equal lines in Geometry, making equal angles with each other, and which are the quotients

* Quantities may be of the same kind though different in quality or affection: thus all straight lines are quantities of the same kind, though they may differ in relative or in absolute position.

† For, if not, let the magnitude of the quantity expressed by $a r$, considered without regard to its affection, be $r(1+c)$: then since $a^2 r$ bears the same relation to $a r$ that $a r$ bears to r, it will follow that the magnitude expressed by $a^2 r$ must be $r(1+c)^2$: similarly, the magnitude of $a^3 r$ will be $r(1+c)^3$, that of $a^4 r$, $r(1+c)^4$ ... and that of $a^n r, \ldots r(1+c)^n$: but $a^n r = r$, and therefore $r = r(1+c)^n$: or in other words, $c = 0$.

of the division of 4, 8, 12 right angles by the number of terms in a period, will be found to present an exact correspondence to the symbolical conditions considered in the last Article, and will enable us to give to them a consistent and complete interpretation.

Let a circle be described with a radius $AB\,(r)$, and let its circumference be supposed to be divided into n equal parts in the points B_1, B_2, $B_3 \ldots B_{n-2}$, B_{n-1}; the radii AB_1, AB_2, AB_3, AB_{n-2}, AB_{n-1}, and AB being drawn to the several points of division: then, if AB be represented in magnitude and position by r, AB_1 may likewise be represented in magnitude and position with respect to AB by ar, AB_2 by $a^2 r$, AB_3 by $a^3 r \ldots$ and AB_{n-1} by $a^{n-1} r$, where a is a base of the period 1, a, a^2, $a^3 \ldots a^{n-1}$.



For, since AB, AB_1, AB_2, AB_3, &c. are equal in magnitude and identical in position *when considered with respect to each other*, Art. 726, it will follow, that if AB_1 can be represented in magnitude and position with respect to AB by $a \times AB$, AB_2 may be equally represented in magnitude and position with respect to AB_1 by $a \times AB_1$, AB_3 by $a \times AB_2, \ldots AB_{n-1}$ by $a \times AB_{n-2}$, and finally AB by $a \times AB_{n-1}$.

Again, if we make $AB = r$, we get

$$AB_1 = a \times AB = a\,r,$$
$$AB_2 = a \times AB_1 = a^2 r,$$
$$AB_3 = a \times AB_2 = a^3 r$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$AB_{n-1} = a \times AB_{n-2} = a^{n-1} r,$$
$$AB \quad = a \times AB_{n-1} = a^n r \text{ or } r,$$

and it will follow, therefore, that the successive lines AB_1, AB_2, $AB_3 \ldots AB_{n-1}$, AB, which are assumed to be severally represented with respect to each other by

$$a \times AB,\ a \times AB_1,\ a \times AB_2 \ldots a \times AB_{n-2},\ a \times AB_{n-1},$$

will be represented, under the same circumstances, with respect to the primitive line AB or r, by the several terms of the period $a\,r$, $a^2 r$, $a^3 r \ldots a^{n-1} r$, $a^n r$, the line represented by the last term

$\alpha^n r$, coinciding with the primitive line or r, in exact conformity with the symbolical condition, which gives $\alpha^n = 1$, and which makes the term $\alpha^n r$ the commencement of a new period, whose successive terms are identical with those of the preceding period, and so on, for ever.

It appears therefore, that if α, which is an imaginary root of $(1)^{\frac{1}{n}}$, be multiplied into r, which represents a given line, then the product of α and r or αr, may represent a line of equal length, making an angle with the primitive line r, which is $\frac{1}{n}$th part of 4, or of some multiple of 4 right angles.

Again, if we should make α^2 the base of the period instead of α, the terms of the new period which arises, or

$$\alpha^2 r,\ \alpha^4 r \dots \alpha^{n-1} r,\ \alpha r,\ \alpha^3 r \dots \alpha^4 r,$$

would severally represent the magnitude and position of the lines AB_2, $AB_4 \dots AB_{n-1}$, AB_1, AB_3, $AB_{n-2} \dots AB$, where each line makes, with that which precedes it in the series, an angle, such as B_2AB, $B_4AB_2 \dots$, which is the double of B_1AB, and which renders it necessary therefore to pass twice round the circumference, or through 8 right angles, before we return to the primitive line AB: and if we should make α^r the base of the period instead of α, the successive lines which its terms would represent would be AB_r, AB_{2r}, $AB_{3r} \dots$, the final term of the period (Art. 721), being AB_{nr}, which coincides with AB, and which we reach for the first time, after passing round the circumference r times, or through $4r$ right angles: and it may be further observed, that the primitive line AB or r, is the first of the n radii AB_1, $AB_2 \dots AB$, which, in this course of circulation, we reach for the second time.

Choice of the base α, which corresponds to the least angle of transfer.

729. We have assumed, in the preceding investigations, α to be the appropriate sign of affection, which when multiplied into AB or r, shall denote the magnitude and position of AB_1 with respect to AB, and therefore α^2 to be the corresponding sign for AB_2, α^3 for AB_3, and α^{n-1} for AB_{n-1}: but inasmuch as α may express any one of the $(n-1)$ roots of $(1)^{\frac{1}{n}}$, which are different from 1 (Art. 708), there is apparently no reason why one of them should enter into the expression which designates the relative position of AB_1 with respect to AB corresponding to the least angle of transfer, in preference to any other: it will be found, however, when we are enabled, in the progress

of our enquiries, to assign *explicitly* the symbolical forms of $(1)^{\frac{1}{n}}$, (Chap. XXXI.) for specific values of n, that every term in this series will possess a *determinate* use and interpretation, as will be partially seen likewise in the particular examples which follow: in the absence, however, of such determinations, we may assume, as we are at liberty to do, a to express that term in the series of roots which corresponds to the least of the angles of position or transfer, or to B_1AB, after which assumption the preceding theory will at once assign to all the other terms of the period their determinate signification.

730. The interpretation, which in the preceding Articles we have applied to the radii of the same circle, which make determinate angles (that is, $\frac{1}{n}$th part of 4 right angles, or of some multiple of 4 right angles) with a primitive radius, are equally applicable to any lines which are equal to them, and which severally make the same angles with the primitive line: for we have elsewhere shewn (Art. 561), that equal lines which have the same relative, but different absolute, positions in space, are expressed by the same symbol with the same sign, whether + or −; and the same reasoning, which was applied to those primitive signs, will be found to be equally applicable to the more general signs which we are now considering.

The preceding interpretations are equally applicable to any equal lines which make the same determinate angles with each other.

731. Again, such interpretations may be indifferently considered either with reference to an operation, or to the result of an operation: that is, we may consider the multiplication of a (a base of $(1)^{\frac{1}{n}}$) into r, as a specific *operation*, whose symbolical result is ar, and whose geometrical result is the transfer of the line denoted by r (AB) through a determinate angle BAB_1: or we may simply consider it as the *result of an operation*, or ar as simply representing the line AB_1, making a determinate angle BAB_1 with the primitive line AB. The *operation* itself is the transfer of the line AB through the determinate angle BAB_1: the *result of the operation* is the line AB_1 having a determinate position with respect to AB, considered without reference to the manner in which it attained it: it is under this second view that we may consider the lines denoted by $a^n r$ and by r as absolutely identical with each other, insomuch as we suppress all reference to the operation or process of transfer by which one line is brought into absolute coincidence with the other.

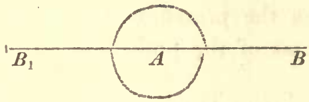
They may be considered either as operations or as results of operations.

We shall now proceed to apply the preceding theory to some examples, in which the symbolical forms of the roots of 1 are explicitly determined.

Application of the preceding theory to the geometrical interpretation of the square roots of 1.

732. The square roots of 1 are 1 and -1: and $a = -1$ (Art. 704), is the corresponding base of the period -1, $(-1)^2$ or -1, 1.

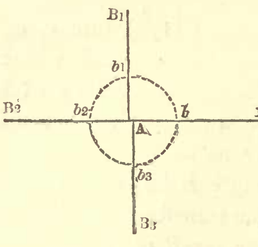
In this case, if r represent AB in magnitude and position, ar or $-r$, will represent AB_1 which makes an angle equal to two right angles ($\frac{1}{2}$ of 4 right angles) with AB, a result which indicates, that in passing from the first to the second position, AB is *transferred* through 2 right angles (Art. 728): the second or last term a^2r of the period is r or AB; and inasmuch as AB_1 is transferred through 2 right angles in passing from the position AB_1 to AB, corresponding to the multiplication of ar by a, it coincides in position and value, as well as in its symbolical expression, with the primitive line AB or r. It may be remarked that the general theory coincides thus far (and they proceed in common no further) with the interpretations given in Art. 728, and those which follow.



Its application to the biquadratic roots of 1.

733. The biquadratic roots of 1, or the values of $(1)^{\frac{1}{4}}$ are 1, -1, $\sqrt{-1}$ and $-\sqrt{-1}$ (Art. 706): if we make $\sqrt{-1}$ (a) (Art. 714), the base, the period will be $\sqrt{-1}$, -1, $-\sqrt{-1}$, 1.

If we denote AB in magnitude and position by r, and if AB_1, AB_2, AB_3, AB make right angles with each other, then it would appear by the principles of the proposition in Art. 728, that AB_1 would be represented in magnitude and position with respect to AB or r by ar or $r\sqrt{-1}$, AB_2 by a^2r or $-r$, AB_3 by a^3r or $-r\sqrt{-1}$, and AB by a^4r or r where the primitive line AB is reproduced both symbolically and geometrically*.



* If we had made $-\sqrt{-1}$ the base, then ar or $-r\sqrt{-1}$ would have denoted AB_3, or the line AB transferred from AB through 3 right angles in the direction b, b_1, b_2, b_3: also, in that case, a^2r or $-r$, would have denoted AB_2 or the line AB_3 transferred through 3 right angles in the direction b_3, b, b_1, b_2: a^3r or $r\sqrt{-1}$ would have denoted AB_1, and a^4r would have denoted AB or r.

It thus appears, that if a line, in a given position, be denoted by r, equal lines making one or three right angles with it, will be symbolically represented by $r\sqrt{-1}$ and $-r\sqrt{-1}$ respectively, a conclusion of fundamental importance in the theory of the interpretation of the results of Algebra when applied to Geometry.

Geometrical interpretation of the sign $\sqrt{-1}$ applied to straight lines.

If we suppose the effect of the multiplication of $\sqrt{-1}$ into r, denoting the line AB, to be its transfer from AB to AB_1 through the right angle BAB_1 in the direction bb_1, the effect of the multiplication by $-\sqrt{-1}$, will be its transfer from AB to AB_3, through the three right angles BAB_1, B_1AB_2, B_2AB_3, in the direction b, b_1, b_2, b_3: if we suppose the directions of the movement of transfer to be reversed, the signs corresponding to the same angles will be changed from + to − and conversely, the angles generated being subject to a similar change, Art. 563.

734. The cubic roots of 1, or the values of $(1)^{\frac{1}{3}}$ (Art. 705) are

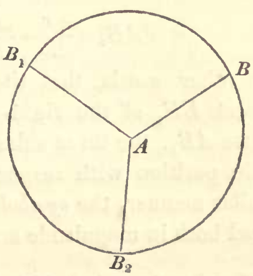
Application to the cube roots of 1.

$$1, \quad \frac{-1+\sqrt{3}\sqrt{-1}}{2} \quad \text{and} \quad \frac{-1-\sqrt{3}\sqrt{-1}}{2}:$$

and if we make $\dfrac{-1+\sqrt{3}\sqrt{-1}}{2}$ the base a, the period will be

$$a,\ a^2,\ a^3, \text{ or } \frac{-1+\sqrt{3}\sqrt{-1}}{2},\ \frac{-1-\sqrt{3}\sqrt{-1}}{2},\ 1.$$

If we draw AB, AB_1, AB_2 dividing the circle into three equal parts, or making angles BAB_1, B_1AB_2, B_2AB with each other which are severally equal to $\frac{1}{3}$rd of 4 right angles, or to 120^0, then if r denote AB in magnitude and position,



$$ar \text{ or } \left(\frac{-1+\sqrt{-3}\sqrt{-1}}{2}\right)r$$

will denote the magnitude and position of AB_1 with respect to AB, and

$$a^2r \text{ or } \left(\frac{-1-\sqrt{3}\sqrt{-1}}{2}\right)r,$$

the magnitude and position of AB_2 with respect to AB: but if we had made $\frac{-1-\sqrt{3}\sqrt{-1}}{2}$ the base, instead of

$$\frac{-1+\sqrt{3}\sqrt{-1}}{2},$$

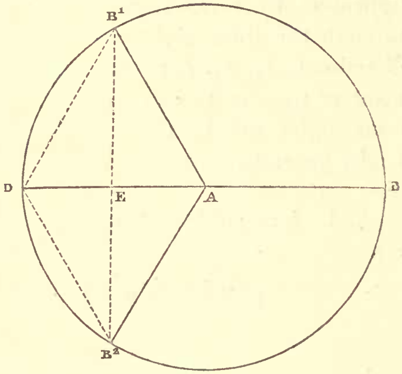
then ar would denote AB_2, a^2r would denote AB_1 and a^3r would denote AB, thus passing twice round the circle before we reach AB.

Sum and difference of two lines when considered with reference to position as well as magnitude.

If we produce BA to D, and join B_1B_2, which bisects AD in E, we shall find

$$B_1E = \frac{\sqrt{3}}{2}.AB^*$$

$$\text{and } EB_2 = \frac{-\sqrt{3}}{2}.AB,$$



since EB_1 and EB_2 being considered merely with reference to each other, have opposite signs, one + and the other −: it will follow, therefore, that if EB_1 be further considered with reference to AB or r, it will be represented both in magnitude and position by $\frac{\sqrt{3}}{2}r\sqrt{-1}$ (Art. 733), and EB_2 by $-\frac{\sqrt{3}}{2}r\sqrt{-1}$: it will follow, therefore, that

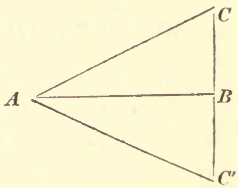
$$AB_1 = \frac{-r}{2} + \frac{\sqrt{3}}{2}r\sqrt{-1} = AE + EB_1$$

$$AB_2 = \frac{-r}{2} - \frac{\sqrt{3}}{2}r\sqrt{-1} = AE - EB_2,$$

or in other words, that the *symbolical sum* of the two sides AE and EB_1 of the right-angled triangle AEB_1 is the hypothenuse AB_1, the three sides being considered in magnitude and also in position with respect to the primitive line AB: and in a similar manner, the *symbolical sum* of AE and EB_2, when considered both in magnitude and position with respect to AB, is the

* For since the angle B_1AD is 60° or $\frac{2}{3}$rd of a right angle, the triangle B_1AD is equilateral, and the perpendicular B_1E bisects the base: it follows likewise that $AB^2 = AE^2 + EB_1^2$, and therefore $EB_1^2 = AB_1^2 - AE^2 = AB^2 - \frac{1}{4}AB^2 = \frac{3}{4}AB^2$, and therefore $EB_1 = \frac{\sqrt{3}}{2}AB$.

hypothenuse AB_2 of the right-angled triangle AEB_2, where EB_2 is drawn in an opposite direction to EB_1, and therefore affected with an opposite sign*: this is a case of a general proposition, which will be established in a subsequent Chapter (XXXI), by which it will appear that if a and $b\sqrt{-1}$ represent two sides AB and BC (where the position of BC is referred to AB) of a right-angled triangle ABC, their *symbolical sum* or $a+b\sqrt{-1}$ will be the hypothenuse AC: and similarly if a and $-b\sqrt{-1}$ or AB and BC' be the two sides of a right-angled triangle ABC', their *symbolical sum* or $a-b\sqrt{-1}$ (or the symbolical *difference* of AB and BC), will be AC'.



Again, it will be found that

$$AB_1+AB_2=\left(-\frac{r}{2}+\frac{\sqrt{3}}{2}r\sqrt{-1}\right)+\left(-\frac{r}{2}-\frac{\sqrt{3}}{2}r\sqrt{-1}\right)=-r=AD,$$

and

$$AB_1-AB_2=\left(-\frac{r}{2}+\frac{\sqrt{3}}{2}r\sqrt{-1}\right)-\left(-\frac{r}{2}-\frac{\sqrt{3}}{2}r\sqrt{-1}\right)$$

$$=\sqrt{3}\,r\sqrt{-1}=2EB_1=B_1B_2:$$

or in other words, the *symbolical sum* of the two sides AB_1 and AB_2 of the rhombus AB_1DB_2 is the diagonal AD which they include, and their *symbolical difference* AB_1-AB_2 is the second diagonal B_1B_2 which is at right angles to the former: the same proposition will be shewn hereafter to be true with respect to the sides and diagonals of any rhombus whatsoever.

It will be found however to be impossible to demonstrate generally these and other important propositions, or to make any extended applications of Symbolical Algebra to Geometry, without the aid of a knowledge of the theory of angles and their measures, and of the various periodical ratios which constitute the science of Trigonometry, or more properly Goniometry, and which we shall proceed to consider at length in the Chapters which immediately follow.

* The lines EB_1 and EB_2, being drawn in opposite directions, are affected with opposite signs, one + and the other −: the application of these signs is independent of the sign $\sqrt{-1}$, which farther indicates that EB_1 is perpendicular to AB with reference to which it is estimated.

CHAPTER XXVI.

ON THE REPRESENTATION AND MEASURES OF ANGLES.

The sciences of Goniometry, Trigonometry, and Polygonometry.

735. THE theory of angles, their measures, and the periodical ratios which determine them, constitutes the science of *Goniometry*, whilst the specific application of some of its results to the determination of the sides, angles, and areas of triangles would be properly termed *Trigonometry*, and to rectilineal figures in general *Polygonometry:* but it has arisen from the associations connected with the progress of our knowledge of these sciences, that the least general of these denominations has anticipated and superseded the adoption of the others, and it is usual to include, under the name of *Trigonometry*, the science of *Goniometry* in its most extended applications.

Angles in Geometry considered with reference to their magnitude only, and not with respect to their mode of generation.

736. Angles are considered in Geometry, as absolute magnitudes only, without any reference to their mode of generation: they may possess every magnitude between *zero* and *two right angles*, which are their limiting values: for lines which contain *no geometrical angle* with each other, and which, in conformity with other views of the generation of angles, make angles with each other equal to zero, or two right angles, or any multiple of 2 right angles, are either parallel to each other, or in the same straight line: and such lines are not distinguished from each other in Geometry, as being drawn in the same or in opposite directions*.

Angles may be conceived to be generated by the revolution of a line round a fixed point and from a given position.

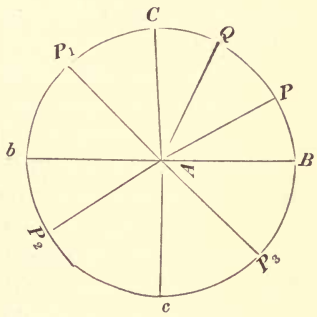
737. As lines, however, may be conceived to be generated by the motion of a point (Art. 558), so likewise angles may be conceived to be generated by the motion or revolution of

* In applying the principles of Symbolical Algebra to the representation of lines, we have been enabled (Arts. 561 and 562) to form two classes of parallel lines (including those which are in the same straight line), according as the lines which form them are drawn in the same or in opposite directions, or according as they form angles with each other, which are an *even* or an *odd* multiple of two right angles, zero being included amongst the former: this classification of parallel lines is not recognized in Geometry, where angles cease to exist at the extreme limits of zero or two right angles.

a line round a fixed point: when viewed with reference to such a mode of generation, and not with respect to their magnitudes merely, angles will be found to be not only capable of indefinite increase, but likewise of affections which may be symbolized by the ordinary signs of algebra.

Relation between arcs and the angles they subtend.

Thus, if a radius AP revolve from the primitive position AB to the position AP, it may be said to pass over or generate the angle BAP, whilst the point P passes over or generates the arc BP: if the movement be continued from AP to AQ, the arc BP will be increased by PQ, and the angle BAP by QAP: and it will follow from a well-known proposition*, that the angles BAP and BAQ will bear to each other the same proportion with the arcs BP and BQ, by which they are subtended in the circumference of the same circle.



Degrees and minutes.

738. If the subtending arc be the quadrant BC, the corresponding angle BAC is a right angle: and if we suppose the quadrant BC to be divided into 90 equal parts, the right angle will be divided by the radii which pass through these points into 90 equal angles, each of which is called a degree: and if the arc subtending a degree be divided into 60 equal parts, each of them will correspond to, or subtend, an angle which is $\frac{1}{60}$th part of a degree, which is called a *minute:* if the arc subtending a minute, be divided into 60 equal parts, each of them will correspond to an angle, which is $\frac{1}{60}$th part of a minute, and is called a *second:* and we may proceed similarly to other inferior units in the sexagesimal scale†, as far as we choose to extend them.

The sexagesimal scale.

* Euclid, Book vi. Prop. 26.

† The sexagesimal division of the circle has prevailed since the time of Ptolemy or the astronomers who preceded him, (see the article "Arithmetic" in the Encyclopædia Metropolitana, p. 401), and is so intimately associated with our habits of thinking and speaking on such subjects, as well as with our astronomical instruments,

Angles thus generated capable of indefinite increase.

739. It thus appears that the angle generated will increase in the same proportion with the corresponding arc which is described or passed over, by the extremity P of the revolving radius AP: and consequently if one of them admit of indefinite increase, so likewise will the other: thus, if P advance to C (Fig. Art. 737), the extremity of the quadrant BC, the angle generated is the right angle BAC or 90^0: if P move onwards to P_1 in the second quadrant of the circle, the corresponding angle BAP_1 is greater than one right angle and less than two, bearing to the right angle BAC the same proportion that the arc BCP_1 bears to the quadrantal arc BC: the revolving radius AP will reach the position Ab, after describing two right angles BAC and CAb, and it is in this sense that the lines AB and Ab which are drawn in opposite directions, are said to make with each other an angle equal to two right angles*: if the motion of AP be continued, it will reach the position AP_2 after describing two right angles together with the angle bAP_2, or an angle equal to $180^0 + bAP_2$†: when it reaches c, at the extremity of the third quadrant, the generating radius has described or passed over three right angles, or 270^0: if the movement be further

instruments, tables, and records, as to make its abandonment somewhat inconvenient and embarrassing: but the superior brevity and uniformity of processes of computation adapted to the decimal scale is tending rapidly to replace the sexagesimal by the *decimal*, or rather by the *centesimal*, division of the degree: thus $13^0.27'.42''$ in the sexagesimal scale is equivalent to $13^0.4633$ nearly in the decimal or to 13 degrees 46 (centesimal) minutes and 33 (centesimal) seconds nearly.

Advantages of the centesimal division of the degree.

The French division of the quadrant.

The French, simultaneously with the establishment of their *Système métrique décimale*, proposed to divide the quadrant into 100 degrees, the centesimal degree into 100 minutes, and the centesimal minute into 100 seconds, and so on, and this division was adopted in the *Mécanique Céleste* of Laplace and other cotemporary scientific works. The change however from the *nonagesimal* to the *centesimal* degree, was attended with no advantages sufficient to compensate for the great sacrifices of tables and records which its adoption rendered necessary, and its use was speedily abandoned, even in France. If the proposed change had been limited to the centesimal division of the nonagesimal degree, it could hardly have failed, when the authority of the great men who proposed it is considered, to have been readily and universally adopted.

Causes of its failure.

* At this point the *geometrical* angle contained by AB and the revolving radius ceases to exist, inasmuch as the lines which contain it are in the same straight line: no regard is paid in Geometry to the directions of lines when considered *per se*.

† The corresponding geometrical angle, which is formed by AB and AP_2 is BAP_2 or $360^0 - (180^0 + bAP_2) = 180^0 - bAP_2$: this is sometimes called the *supplement* of BAP_2 to 360^0.

continued to P_3, in the fourth quadrant, the corresponding angle generated is $270^0 + c\,AP_3$, and when P reaches B, or the point from which it started, after describing the entire circumference, the corresponding angle generated is 4 right angles, or 360^0. If we proceed onward for a second revolution, we must add 360^0 or 4 right angles to the angles generated at the corresponding points in the first revolution: and similarly for every additional revolution of the revolving or generating radius, adding, in the $(n+1)^{th}$ revolution, $n \times 360^0$, or $4n$ right angles to the symbolical or numerical value of the angle in the first.

740. The angles which are thus generated, and which are capable, like other magnitudes, of indefinite increase, may be termed *goniometrical*, as distinguished from *geometrical*, angles: thus, if the *geometrical* angle BAP, which AP makes with AB, be called A, the series of *goniometrical* angles A, $360^0 + A$, $2 \times 360^0 + A$, $3 \times 360^0 + A$, $\ldots\, n \times 360^0 + A$, will severally designate the same position of AP with respect to AB, implying the additional condition that this position had been attained in the 1st, 2nd, 3rd, 4th, ... $(n+1)^{th}$ revolution of the generating radius AP. Goniometrical and geometrical angles.

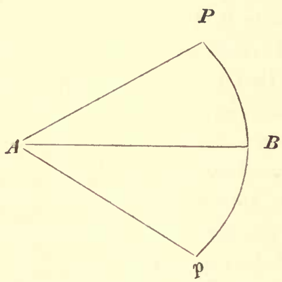
It may be observed that *goniometrical* angles, which are less than 180^0, or two right angles, coincide both in magnitude and in the position which they express, with the corresponding *geometrical* angles: but that *goniometrical* angles which are greater than 180^0, or two right angles, are different in magnitude, but coincident, in the position which they express, with the corresponding *geometrical* angles, the geometrical angle being formed above or below the line BAb, according as the goniometrical angle contains an *even* or an *odd* multiple of 180^0*. In what cases they coincide.

741. We recognize no distinction in Geometry, where the magnitudes of angles alone are considered, between those which are formed upon different sides of the primitive line AB, or more generally between those which are generated by movements of revolution in opposite directions: but in Symbolical Algebra, where some of the affections of magnitudes, as well as the magnitudes themselves, are capable of being considered, Angles generated by opposite movements of revolution have different signs.

* If the goniometrical angle be expressed by $2n \times 180^0 + A$, where A is less than two right angles, the geometrical angle corresponding is A, and is above the line $BA\,b$: if the goniometrical angle be expressed by $(2n+1)\,180^0 + A$, the corresponding geometrical angle is $180^0 - A$, and is formed below the line $BA\,b$.

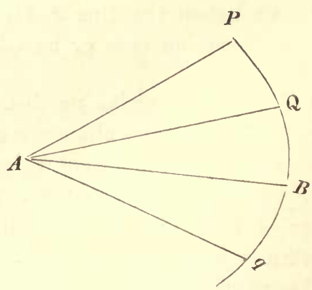
we shall find that such angles, when expressed symbolically, will differ in their signs, one series being positive and the other negative, or conversely.

For we have shewn in Art. 563, that if an arc BP described by the movement of a point from B to P, be represented by a or $+a$, an equal arc Bp, described by a corresponding movement from B to p, in an opposite direction, would be correctly symbolized by $-a$: and the same reasoning which is there applied, would equally shew, that if an angle BAP, generated by a radius revolving from AB to AP be represented by A or $+A$, an



equal angle BAp generated by the radius revolving in the opposite direction, or from AB to Ap, would be correctly represented by $-A$, and conversely: and generally, whenever angles are generated by movements of revolution which are in opposite directions, they will be correctly expressed by symbols representing their absolute magnitudes, with different signs, one $+$ and the other $-$*.

* The process of reasoning applicable to this case is as follows. Let A and B represent the magnitudes of the angles BAP and PAQ as used in Arithmetical Algebra, and therefore $A-B$ will express their difference, which is the angle BAQ: but, in Symbolical Algebra, A and B, which are general in form, are general likewise in value, and therefore we may suppose B to be greater as well as less than A. Let us suppose $B = A + C$, where C represents the magnitude of the angle BAq or the excess of the angle PAq above BAP: we thus get $BAq = A - B = A - (A + C) = -C$: or in other words, an angle formed on the opposite side of AB to that on



which the angle $+A$ is formed and which is generated by a movement in an opposite direction, is represented by a symbol expressing its absolute magnitude, but with a different sign.

742. In considering, therefore, the different modes in which a generating or revolving radius may reach a given position with respect to a primitive line, we must have regard not merely to the quantity, but to the direction of the movement of revolution by which we pass from one to the other. The most general expression of the same position by means of goniometrical angles.

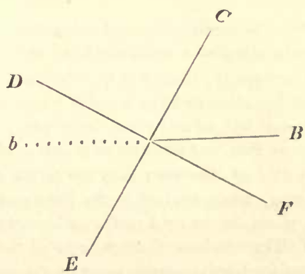
Thus, if AB be the primitive position of the generating radius (Fig. in Art. 741), we may arrive at the position AP, making an angle A, less than 180^0, with AB, after describing the series of *goniometrical* angles, A, $360^0 + A$, $720^0 + A, \ldots n \times 360^0 + A$, and reaching AP once, twice, thrice, ... $(1 + n)$ times: but if we reverse the movement, generating negative angles, the corresponding series of *goniometrical* angles will be $-360^0 + A$, $-720^0 + A$, $-1080^0 + A \ldots - n \times 360^0 + A$: and similarly if $-A$ be the angle which Ap makes with AB, and therefore on the opposite side to that on which the angle A is formed, then the corresponding series of *goniometrical* angles formed by direct and reversed movements will be

$$360^0 - A,\ 720^0 - A,\ 1080^0 - A, \ldots (1 + n) \times 360^0 - A,$$

$$\text{and } -A,\ -360^0 - A,\ -720^0 - A, \ldots -n \times 360^0 - A:$$

as far, therefore, as regards the position of one line with respect to another, which makes with the primitive line an angle $\pm A$, when A is less than 180^0, it will be equally expressed by all the *goniometrical* angles which are included in the formula $\pm n \times 360^0 + A$ in one case, and $\pm n \times 360^0 - A$ in the other.

743. And conversely, we may pass from any goniometrical angle to the corresponding geometrical angle, (which will be *symbolically* expressed by a positive or negative value of A, according as it is formed above or below the primitive line AB), by adding to it, or by subtracting from it, such a multiple of 360^0 as may leave a positive or a negative residual which is less than 180^0, or than the extreme limit of geometrical angles: thus the *goniometrical* angle 427^0 denotes a position identical with that Transition from the goniometrical to the corresponding geometrical angle.



denoted by the geometrical angle $427^0 - 360^0$ or 67^0 or the angle BAC in the annexed figure: the goniometrical angle 534^0 is identical with the geometrical angle $534^0 - 360^0$ or 174^0 (BAD): the goniometrical angle $597^0.11'.44''$ corresponds to

$$597^0.11'.44'' - 2 \times 360^0 = - 122^0.47'.16'',$$

or to the geometrical angle $122^0.48'.16''$, or BAE formed below the primitive line AB, or bAB: the goniometrical angle -1104^0 corresponds to the symbolical angle $-1104^0 + 3 \times 360^0 = -24^0$, which is equivalent to the geometrical angle 24^0, or BAF formed likewise below* AB: and similarly in other cases.

Definition of a measure.

744. One variable quantity may be said to measure another of a *different kind*, if it increases or diminishes indefinitely in the same proportion with it†.

It will follow as a necessary consequence of this *definition*, that if the *measure* is capable of continuous and indefinite increase, the *quantity measured* must possess the same property.

Thus, arcs of the same circle are *measures* of the *goniometrical* angles which correspond to them, inasmuch as they increase continuously in the same proportion (Art. 737) from zero to infinity: but such arcs cannot properly be considered as measures of the corresponding *geometrical* angles, inasmuch as whilst the arcs increase indefinitely, the corresponding values of the angles are *periodical*, being always included between the limits of zero and two right angles.

Measures determinate or indeterminate.

745. Measures are said to be *determinate* or *indeterminate*, according as a determinate or indeterminate magnitude of the

* The terms *above* and *below* are relative, being applied to angles formed on opposite sides of a primitive line, one of which is *positive* and the other *negative*, and conversely, though it is indifferent in what order they are taken: such terms, as well as other relative terms of a similar class, such as *high* and *low*, *up* and *down*, *right* and *left*, *to* and *from*, have generally some reference, not merely to a zero point or line, but likewise to relations of position, which are real with reference to the writer or observer: thus the terms *above* and *below*, are used in their ordinary meaning, when applied to the lines and angles formed in the figure given in the text, when viewed by a reader in his ordinary position.

† The relations of magnitudes of the same kind are numerical only, and may be replaced by *commensurable* or *incommensurable* numbers (Art. 165): it is the common unit of magnitude expressed by one of these numbers, when they are *commensurable*, which is said to measure the magnitudes which they express, a notion of a *measure* which is quite distinct from that which is considered in the text.

measure corresponds to a determinate magnitude of the quantity measured.

Thus an arc of a *determinate* length in a circle, whose radius is *given* or *determinate*, corresponds to a *determinate* goniometrical angle, and it is therefore a *determinate* measure of such angles: but if the radius of the circle in which the arcs are reckoned, is not given or is *indeterminate*, then a *determinate* value of the arc will not correspond to a *given* or *determinate* angle, and it is therefore no longer a *determinate* measure of such angles: it follows, therefore, that arcs are determinate measures of the corresponding angles, when the radius of the circle in which such arcs are taken is determinate or given, and not otherwise.

746. If, however, the arc alone be an *indeterminate*, the ratio of the arc to the radius of the circle on which it is taken, will be a *determinate*, measure of the corresponding *goniometrical* angle, whatever that radius may be*: it is this ratio which is assumed as the measure of angles in the articles which follow.

The arc divided by the radius of the circle in which it is taken is a determinate measure of the corresponding angle.

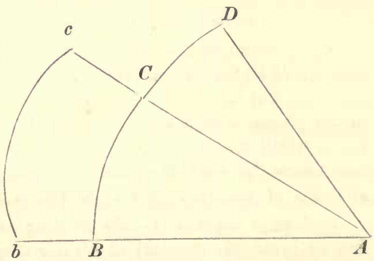
747. The ratio of the arc to the radius is numerical, and if its value can be determined for the whole or any definite portion of the circumference, it can be determined by a simple proportion for any other.

If this ratio is given in one case it is given in all others.

* For if the arcs BC and bc subtend the same angle A in the circles whose radii are AB and Ab respectively, then by a well known property of the circle, we have $\frac{BC}{AB} = \frac{bc}{Ab}$: and if we take the angle BAD in the circle whose radius is AB and the angle bAc in the circle whose radius is Ab, then we find

$$\angle BAC : \angle BAD :: BC : BD$$

$$:: \frac{BC}{AB} : \frac{BD}{AB} :: \frac{bc}{Ab} : \frac{BD}{AB}:$$



and since these angles increase in the same proportion with the corresponding ratios, it follows that $\frac{BC}{AB}$ or its equivalent $\frac{bc}{Ab}$ is the proper and determinate measure of the corresponding angle.

The ratio of the circumference to the radius expressed by 2π.

Thus, the ratio of the whole circumference to the diameter, though they are not commensurable, has been approximately determined by various methods, some of which will be considered hereafter. This ratio, which very frequently presents itself in analytical formulæ, is approximately expressed by the number 3.14159, and is also generally represented by the symbol π: the ratio of the circumference to the radius, which is the double of the ratio of the circumference to the diameter, is therefore expressed by 2π*.

The measure of 90^0 is expressed by $\frac{\pi}{2}$.

The ratio of a quadrant to the radius, which is the *measure* of a right angle or 90^0, is one-fourth part of the ratio of the circumference to the radius, and is therefore expressed by $\frac{\pi}{2}$.

Unity is the measure of an angle of $57^0.17'.45''$, and $.0017453$ is the measure of an angle of 1^0.

If it be required to determine the angle whose measure is 1, or, in other words, the arc whose length is equal to the radius, it will be found to be $57^0.2958$ or $57^0.17'.45''$ nearly: if the measure of an angle of one degree be required, it will be found to be $\frac{\pi}{180}$ or $.001745$ nearly: there exists therefore no simple numerical relation between the unit of measures and the unit of the angles which they measure.

Angles and their measures are very often designated by the same symbols and the same denominations.

748. But though there exists no simple relation between angles and their measures†, by which they may be converted

* Archimedes, in his Κύκλου Μέτρησις, assigned $\frac{22}{7}$ as an approximate value of π, which differs in excess from its true value by less than $\frac{1}{800}$th part of the diameter: it is this value, which is generally used by workmen, in their estimation of circular work. Peter Metius, by a similar process, found the remarkable ratio $\frac{355}{113}$, which is correct to 5 places of decimals, and which may be very easily remembered, by observing that if we write each of the three first odd digits twice in succession, as in the number 113355, the three last digits form its numerator and the three first its denominator. Later researches have assigned its value correctly to 208 places of decimals, a prodigious approximation, which is effected by processes which are extremely simple and expeditious, and well calculated to shew how much the most complicated calculations may be shortened by a judicious selection of formulæ. See Phil. Trans. 1841, p. 281.

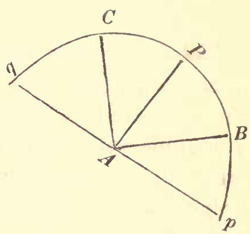
† A right angle is capable of being determined geometrically, and is therefore an invariable standard of angular magnitude, to which all other angles are referrible numerically, and some also geometrically: if we should call a right angle *unity*, and its centesimal subdivisions grades, minutes, seconds, we should have the French division of the quadrant, or of the right angle, with its scale of units increased one hundred-fold.

There is no geometrical mode of determining an absolute unit of any other magnitude. Thus a linear unit must be sought for in some assumed standard,

such

promptly and easily into each other, yet in the case of certain periodical ratios, which will be considered in the next Chapter, which are equally determined by them, it is usual to represent them by common symbols, and to call them by common denominations: thus, $\frac{\pi}{2}$ is considered equally as the representative of a right angle, and its measure: and similarly θ or ϕ (which are symbols very generally employed for such purposes) and any other symbols are applied indifferently to designate both angles or their measures.

749. Upon the same principle, if θ is used to denote either an angle or the measure of an angle, $\frac{\pi}{2}-\theta$ is used to denote its complement, or the measure of its complement to 90°: or, in other words, if θ denotes the angle or the measure of the angle, BAP, then $\frac{\pi}{2}-\theta$ is used to denote the angle as well as the measure of the angle, PAC. Usual mode of denoting an angle and its complement.



750. In speaking of geometrical angles and their *complements*, we assume them both to be less than 90°: but if θ denotes any *goniometrical* angle whatsoever, or its measure, whether greater or less than 90°, we still continue to apply the same term *complement* to $\frac{\pi}{2}-\theta$, though it may no longer possess its ordinary geometrical meaning. Extended application of the term complement.

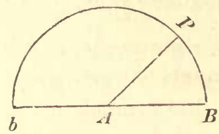
Thus if $\theta = 120^{\circ}$ or BAq, then $\frac{\pi}{2}-\theta = -30^{\circ}$, or BAp (Art. 741); and if $\theta = -30^{\circ}$, or BAp, then $\frac{\pi}{2}-\theta = 120^{\circ}$, or BAq: it follows, therefore, in conformity with the principles of Symbolical Algebra, that the meaning of the term *complement*, when no longer

such as the English standard yard (which no longer exists), or by reference to some invariable standard in nature, such as the length of a pendulum vibrating seconds in *vacuo* in a given latitude at a given height above the sea, or to the length of a quadrant, or other definite part of the earth's meridian, such as the *mètre* of France: the same remark applies to the units of time, force, and all other *physico-mathematical* units.

geometrical, is interpreted altogether with reference to the conditions which its symbolical representative is required to satisfy.

Usual mode of denoting the supplement of an angle.

751. In a similar manner, if θ denote the angle BAP, or its measure, then $\pi - \theta$ is used to denote its supplement $180^0 - \theta$ (PAb), or its measure.



In speaking of geometrical angles and their supplements, we suppose them both to be less than 180^0: but $\pi - \theta$ continues to be called the supplement of θ, when θ is any *goniometrical* angle whatsoever.

Various measures corresponding to the same geometrical angle.

752. The angle $\pm\theta$, or its measure, corresponds to the same geometrical angle with the measure $\pm 2n\pi \pm \theta$, where n is any whole number: in other words, if we merely regard the position of one line with respect to another, we may add to, or subtract from, its measure, any multiple of 2π, or of the measure of 4 right angles: this conclusion follows immediately from Art. 742.

In a similar manner, $\frac{\pi}{2} \pm \theta$ corresponds to the same geometrical angle with $\pm 2n\pi + \frac{\pi}{2} \pm \theta$ or $\frac{(\pm 4n + 1)\pi}{2} \pm \theta$: $\pi \pm \theta$ corresponds to the same geometrical angle with $\pm 2n\pi + \pi \pm \theta$, or $(\pm 2n + 1)\pi \pm \theta$: and $\frac{3\pi}{2} \pm \theta$ corresponds to the same geometrical angle with $\pm 2n\pi + \frac{3\pi}{2} \pm \theta$ or $\frac{(\pm 4n + 3)\pi}{2} \pm \theta$.

These equivalent measures, corresponding to the same geometrical angle, lead to the most important consequences, which will be more particularly considered in the subsequent Chapters.

Great importance of the fundamental propositions in Trigonometry.

753. The propositions in this and the following Chapter will be found to constitute the *grammar*, as it were, of the language of Trigonometry, and will be made the foundation of a very extensive analytical theory, admitting of the most varied and important applications to every branch of mathematical and physical science. It is for this reason that the relations and signs of affection of goniometrical angles, and their measures, and of the periodical ratios (Chap. XXVI.) which determine them, minute

and unimportant as some of them may at first sight appear, will lead to consequences of very great interest and value*: they cannot therefore be too carefully studied and remembered.

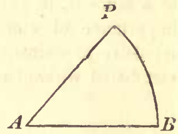
* The whole theory of music is dependent upon the properties and conversions of a very small number of numerical ratios: yet how vast and complicated is the superstructure which is raised upon them! It should be the first lesson of a student, in every branch of science, not to form his own estimate of the importance of elementary views and propositions, which are very frequently repulsive or uninteresting, and such as cannot be thoroughly mastered and remembered without a great sacrifice of time and labour.

CHAPTER XXVII.

ON THE THEORY OF THE SINES AND COSINES OF ANGLES.

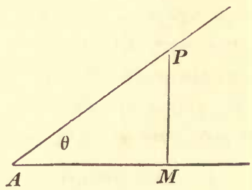
Ratios which determine, but do not measure, angles.

754. An angle, such as BAP, is determined by its measure, which is $\frac{BP}{AB}$ (Art. 746): but there are other ratios which equally *determine* an angle, though they do not *measure* it, possessing properties of very great importance, which we shall now proceed to consider.

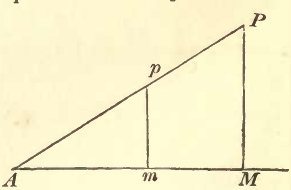


Definition of the sine and cosine of an angle.

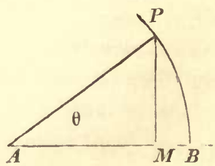
If from any point whatsoever P, in one of the lines containing the angle MAP or θ, we draw a perpendicular PM upon the other, we shall form a right-angled triangle MAP, of which one side PM is opposite, and the other adjacent, to the angle θ: then the ratio $\frac{PM}{AP}$ is called the *sine*, and the ratio $\frac{AM}{AP}$ is called the *cosine* of the angle PAM or θ: and inasmuch as these ratios remain unaltered whatever be the distance of the point P from A*, they are said to determine the angle, inasmuch as a definite value of the *sine* or *cosine* determines a definite value of the corresponding angle†.



* For if we take any other point *whatsoever* p in AP or in AP produced, and draw pm perpendicular to AM or to AM produced, we shall find, from a well-known property of similar triangles, that $\frac{pm}{Ap} = \frac{PM}{AP}$, and $\frac{Am}{Ap} = \frac{AM}{AP}$, and therefore the ratios called *sine* and *cosine* are always the *same* for the *same* angle: if therefore the angle be given, they will possess a determinate value.



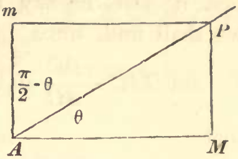
† It was formerly the practice to define the *sine* and *cosine* as *lines*, and not as *ratios:* thus if we take an arc BP subtending an angle BAP (θ) in a circle whose radius is AB, and if from the extremity P of the arc we draw PM a perpendicular upon the radius AB passing through its beginning, then PM was defined to be the *sine* of the angle BAP, and AM its *cosine:* it followed from this definition, that the sine



and

755. If we form the right angle MAm, drawing Pm perpendicular to Am, then the angles PAM and PAm will be *complementary* to each other (Art. 749); and if one of them PAM be denoted by θ, the other PAm may be denoted by $\frac{\pi}{2} - \theta$: and it will follow from the definitions of the *sine* and *cosine* which are given in the last Article, that $\frac{Pm}{AP}$ is the *sine*, and $\frac{Am}{AP}$ is the *cosine* of the angle PAm, or $\frac{\pi}{2} - \theta$: but since $Pm = AM$ and $PM = Am$, it follows that

The sine of an angle is the cosine of its complement.



$$\frac{Pm}{AP} = \frac{AM}{AP} \text{ and } \frac{Am}{AP} = \frac{PM}{AP},$$

and therefore the *cosine* of the angle $\frac{\pi}{2} - \theta$ is the *sine* of the angle θ, and the *sine* of the angle $\frac{\pi}{2} - \theta$ is the *cosine* of the angle θ.

It consequently appears that the *cosine* of an angle is the *sine* of its complement, and conversely, a relation of values which is indicated by the composition of the term *cosine.*

756. It is usual to denote the sine and cosine of an angle θ by the abbreviated expressions $\sin\theta$ and $\cos\theta$*: and the proposition in the last Article would, therefore, be symbolically expressed by the equations

Usual mode of denoting the sine and cosine of θ.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad (1),$$

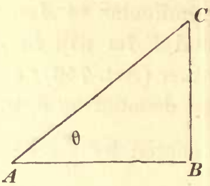
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad (2).$$

and cosine did not *determine* the angle when the radius was not given, and no trigonometrical formula was complete unless the radius (when not assumed to be 1) was introduced as one of its elements: the definition in the text, which is now exclusively used, (see Hymers's Trigonometry, Cambridge, 1837), was first formally introduced into my Algebra, (Cambridge, 1830), and afterwards into a "Syllabus of a Course of Lectures on Trigonometry," (Cambridge, 1833).

* It was Euler who first introduced this notation, and thus created the proper science of Goniometry. See Report on certain Branches of Analysis, page 289, in the Reports of the British Association for 1833.

Expressions for the sides of a right-angled triangle in terms of the hypothenuse, and of the sine and cosine of the angle at the base.

757. Again, it follows, from the definitions given in Art. 754, that if ABC be any right-angled triangle, we shall find, since



$$\sin BAC = \frac{BC}{AC} \text{ and } \cos BAC = \frac{AB}{AC},$$

that

$$BC = AC \sin BAC \text{ and } AB = AC \cos BAC:$$

if, therefore, a, b, c be the hypothenuse, base, and perpendicular of any right-angled triangle, and θ the angle adjacent to the base, we shall get $b = a \cos\theta$ and $c = a \sin\theta$: these expressions will be found to be of very great importance in the application of Algebra (including Trigonometry as one of its branches) to Geometry.

Fundamental relation of the sine and cosine.

758. Inasmuch as $\sin\theta = \frac{PM}{AP}$ (Fig. Art. 754), and $\cos\theta = \frac{AM}{AP}$, it follows that

$$\sin^2\theta + \cos^2\theta = \frac{PM^2 + AM^2}{AP^2} = \frac{AP^2}{AP^2} = 1:$$

for in the right-angled triangle, PAM, we find, by Euclid, Lib. I. Prop. 47, that

$$PM^2 + AM^2 = AP^2.$$

Expression for the sine in terms of the cosine, and conversely.

759. The equation

$$\sin^2\theta + \cos^2\theta = 1 \qquad (3),$$

defines the fundamental relation between $\sin\theta$ and $\cos\theta$, and enables us to express one of them in terms of the other: we thus get

$$\sin\theta = \sqrt{(1 - \cos^2\theta)} \qquad (4),$$

$$\cos\theta = \sqrt{(1 - \sin^2\theta)} \qquad (5).$$

The limits of their values and the signs which affect them.

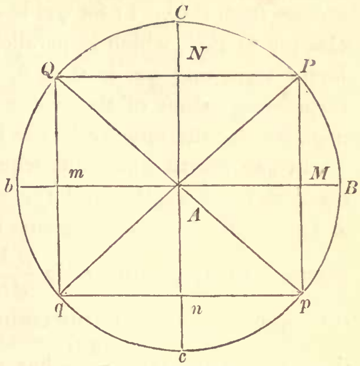
It will appear, from the examination of these expressions, that the limits of the arithmetical values of $\sin\theta$ and $\cos\theta$ are 0 and 1, and that whilst one of them increases, the other diminishes: that the limits of their *symbolical* values are 1 and -1, the fundamental equation recognizing no distinction between the positive and negative values of $\sin\theta$ and $\cos\theta$: and that for every value of $\cos\theta$ there are two values of $\sin\theta$, and conversely, which differ from each other in their signs only, one being positive and the other negative.

Detailed examination of these changes.

760. The course and succession of these changes, however, will appear much more distinctly, from tracing them in detail

through four right angles, by the aid of their geometrical definition (Art. 754) and of the principles of interpretation established in Art. 558, and those which follow.

In the first quadrant, the value of $\sin\theta$ increases from 0 to 1: for $\sin\theta = \dfrac{PM}{AB}$ (Art. 754) is 0 when $\theta = \dfrac{PB}{AB}$ is 0: and when θ is 90°, then PM becomes CA, and



$$\sin\theta = \frac{AC}{AB} = 1:$$

the corresponding values of the cosine diminish from 1 to 0: for if $\theta = 0$, then $AM = AB$, and $\cos\theta = \dfrac{AB}{AB} = 1$: and if $\theta = 90^\circ$, then $AM = 0$, and therefore

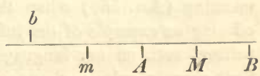
$$\cos\theta = \frac{0}{AB} = 0.$$

In the second quadrant, the values of $\sin\theta$ decrease from 1 to 0: for if θ represent the angle BAQ, then $\sin\theta$ is 1, when Q is at C, and 0 when Q is at b, where Qm is 0: the corresponding values of the cosine change from 0 to -1: for

$$\cos\theta = \cos BAQ = \frac{Am}{AB},$$

which is 0 at C, and equal to -1 at b, where Am coincides with Ab: for since AM and Am are drawn in opposite directions from A, it follows that if $\dfrac{AM}{AB}$ be positive, then $\dfrac{Am}{AB}$ is negative: the cosine in the second quadrant, therefore, increases from 0 to -1*.

* If we speak of magnitudes independently of their signs, we may say that a quantity increases from 0 to -1 with as much propriety as that it increases from 0 to 1: for the signs $+$ and $-$, used as signs of affection, express qualities or specific conditions of existence of magnitudes, and not the magnitudes themselves: thus the line generated by the points M or m moving from A to B in one case, and from A to b in the other, increases equally by its movement through equal spaces, though it is *positive* in one case, and *negative* in the other: and if we subtract the same number repeatedly from another, as 1 from 4, we get a series of remainders



$$3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3,$$

which

In the third quadrant, where the goniometrical angle exceeds two right angles, and where the corresponding geometrical angle (Art. 740) BAq is negative, the sine $\frac{qm}{AB}$ is negative, and increases from 0 to -1: for qm is drawn in an opposite direction to Qm (or to PM, which is parallel to it), and has therefore a different sign, and qm is 0 at b, and equal to Ac at c: the corresponding values of the cosine diminish from -1 to 0, and are negative, for the same reason as in the second quadrant.

In the fourth quadrant, where the goniometrical angle exceeds 3 right angles, and the corresponding geometrical angle BAp is still negative, the sine is also negative, and diminishes from -1 to 0; for the ratio $\frac{pM}{AB}$ is -1 at c, and 0 at B: the corresponding values of the cosine are positive, and increase from 0 to 1: for the ratio $\frac{AM}{AB}$ has precisely the same values as for the equal and positive angles in the first quadrant.

The sine and cosine of goniometrical angles which differ by multiples of 2π *only* are the same as of those of the corresponding geometrical angles.

761. The *same position* of one line with respect to another, or primitive line, and therefore the same geometrical angle θ, (whether positive or negative, less or greater than 90^0), and consequently *the same sine and cosine** will correspond to the series of goniometrical angles θ, $\pm 2\pi + \theta$, $\pm 4\pi + \theta$, $\ldots \pm 2n\pi + \theta$, which differ from each other by multiples of 2π only (Art. 752): and this conclusion, which will be found to lead to the most important consequences, is symbolically expressed by the equations

$$\sin\theta = \sin\{\pm 2n\pi + \theta\} \qquad (7),$$

$$\cos\theta = \cos\{\pm 2n\pi + \theta\} \qquad (8).$$

There are some other general consequences which are easily deducible from the geometrical definition of the sine and cosine

which increase both ways from 0: but it is sometimes very *incorrectly* said that -1 is less than 0, forgetting that the operation called *subtraction* changes its meaning (Art. 555) when the subtrahend is greater than the minuend: thus offering an example of the influence of terms, in their ordinary acceptation, prevailing, both on our language and our conclusions, when the circumstances of their usage require a change of signification.

* For the sine and cosine are altogether dependent, as appears by the definitions in Art. 754, upon the geometrical angle: all goniometrical angles, therefore, which correspond to the same geometrical angle, have the same sine and cosine.

of an angle, which are not less important than those which we have already considered, and which we shall now proceed to notice.

762. The sines of angles, which are equal in magnitude, but different in sign, are also equal in magnitude but different in sign: but the cosines of angles which are equal in magnitude but different in sign, are identical both in magnitude and sign: or, if expressed in symbolical language, The relations of the sines and cosines of positive and negative angles.

$$\sin - \theta = - \sin \theta \qquad (9),$$
$$\cos - \theta = \cos \theta \qquad (10).$$

For, BAP (Fig. in the last Article), and BAp are equal angles, but with different signs (Art. 741): and their sines $\frac{PM}{AB}$ and $\frac{pM}{AB}$ are equal in magnitude, but different in sign: but the same cosine $\frac{AM}{AB}$ is common to both of them.

The same remark applies to the angles BAQ and BAq, which are greater than right angles.

These equations, combined with the fundamental equation
$$\sin^2\theta + \cos^2\theta = 1,$$
completely determine the sine and cosine of an angle and their relations to each other.

763. The sine of an angle is identical with the sine of its supplement (Art. 751), both in magnitude and sign: but the cosine of an angle is equal to the cosine of its supplement in magnitude, but differs from it in sign. The relations of the sines and cosines of angles and their supplements.

For, if the angle BAQ (Fig. in Art. 760), be the supplement of the angle BAP, their sines $\frac{PM}{AB}$ and $\frac{Qm}{AB}$ are equal in magnitude, since $PM = Qm$; and they have the same sign, since PM and Qm are parallel and estimated in the same direction with respect to the line BAb: but the cosine of the angle BAP or $\frac{AM}{AB}$ is equal in magnitude to the cosine of its supplement BAQ, which is $\frac{Am}{AB}$, but differs from it in sign, since AM and Am are drawn in opposite directions from A.

The symbolical enunciation of this proposition is
$$\sin(\pi - \theta) = \sin\theta \qquad (11),$$
$$\cos(\pi - \theta) = -\cos\theta \qquad (12).$$

The relations of the sines and cosines of $\frac{\pi}{2}+\theta$ and θ.

764. It follows, from the proposition in the last Article, that

$$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta \qquad (13),$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta \qquad (14).$$

For $\frac{\pi}{2}+\theta$ is the supplement of $\frac{\pi}{2}-\theta$, and therefore

$$\sin\left(\frac{\pi}{2}+\theta\right)=\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad \text{(Art. 755)},$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\cos\left(\frac{\pi}{2}-\theta\right)=-\sin\theta \quad \text{(Art. 755)}.$$

The same conclusion may be deduced *geometrically* from the figure given in Art. 760.

The relations of the sine and cosine of $\pi+\theta$ and θ.

765. Again, since the sines and cosines of all goniometrical angles, which differ only by multiples of 2π or 360°, are identical with each other (Art. 761), it follows that

$$\sin(\pi+\theta)=-\sin(\pi-\theta)=-\sin\theta \qquad (15),$$

$$\cos(\pi+\theta)=\cos(\pi-\theta)=-\cos\theta \qquad (16).$$

For $\sin(\pi+\theta)=\sin(\pi+\theta-2\pi)=\sin-(\pi-\theta)=-\sin(\pi-\theta)$ (Art. 763), $=-\sin\theta$: and

$$\cos(\pi+\theta)=\cos(\pi+\theta-2\pi)=\cos-(\pi-\theta)=\cos(\pi-\theta) \text{ (Art. 763)},$$
$$=-\cos\theta.$$

Other similar relations.

The same principle will enable us to convert many other expressions of a similar kind into others which are equivalent to them.

Thus, $\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta \qquad (17).$

$$\text{For } \sin\left(\frac{3\pi}{2}-\theta\right)=\sin\left(\frac{3\pi}{2}-\theta-2\pi\right)=\sin-\left(\frac{\pi}{2}+\theta\right)$$

$$=-\sin\left(\frac{\pi}{2}+\theta\right)=-\cos\theta. \quad \text{(Art. 764)}.$$

Similarly, we find

$$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta \qquad (18),$$

$$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta \qquad (19),$$

$$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta \qquad (20).$$

Similarly, if the goniometrical angle, whose sine or cosine is required, exceeds 2π or 360^0, it may be immediately reduced to one of the preceding cases by the subtraction or addition of the greatest multiple of 2π, which it contains. Thus

$$\sin(2\pi + \theta) = \sin\theta,$$

$$\sin\left(\frac{5\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta,$$

$$\sin(3\pi + \theta) = \sin(\pi + \theta) = -\sin\theta,$$

$$\sin\left(\frac{7\pi}{2} + \theta\right) = \sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta,$$

$$\cos(2\pi + \theta) = \cos\theta,$$

$$\cos\left(\frac{5\pi}{2} + \theta\right) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta,$$

$$\cos(3\pi + \theta) = \cos(\pi + \theta) = -\cos\theta,$$

$$\cos\left(\frac{7\pi}{2} + \theta\right) = \cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta.$$

Angles whose sines and cosines admit of a simple numerical expression.

766. The general methods which are adopted for determining and registering in tables the numerical values of the sines and cosines of every angle, expressed in degrees and minutes, between 0 and 90^0, or in other words, *of forming a canon of sines and cosines,* will form the subject of a subsequent Chapter (XXIX): but there are some angles, which are aliquot parts of a right angle, whose sines and cosines admit of a very simple and immediate numerical expression, by the aid of their geometrical properties, and which it will be found to be very useful for the student to remember: the following are examples.

The sine and cosine of 45^0 or $\frac{\pi}{4}$.

The sine and cosine of 45^0 are equal to each other and to $\frac{1}{\sqrt{2}}$.

For $\sin 45^0 = \cos(90^0 - 45^0)$ (Art. 755), $= \cos 45$: and since

$$\sin^2 45^0 + \cos^2 45^0 = 1 \quad \text{(Art. 758)},$$

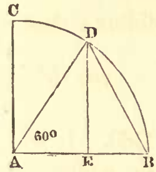
we get $2\sin^2 45^0 = 1$, and therefore $\sin 45^0 = \frac{1}{\sqrt{2}} = \cos 45^0$: the positive sign is taken, since 45^0 is less than $\frac{\pi}{2}$ (Art. 760).

The same conclusion follows immediately from reference to the Figure in Art. 760.

The sines and cosines of 60^0 and 30^0, or of $\frac{\pi}{3}$ and $\frac{\pi}{6}$.

767. The sine of 60^0 or the cosine of 30^0 is $\frac{\sqrt{3}}{2}$: and the cosine of 60^0 or the sine of 30^0 is $\frac{1}{2}$.

For if, upon the base AB we describe the equilateral triangle DAB, the point D will be found upon the circumference of the circle and the angle BAD will be one third of two right angles or 60^0: and if DE be drawn perpendicular to AB, it will bisect it in E: it follows therefore that



$$\cos BAD = \cos 60^0 = \frac{AE}{AB} = \frac{1}{2} = \sin 30^0:$$

and inasmuch as (Art. 758),

$$\cos^2 60^0 + \sin^2 60^0 = 1,$$

we get, by replacing $\cos^2 60^0$ by $\frac{1}{4}$,

$$\frac{1}{4} + \sin^2 60^0 = 1,$$

$$\text{or } \sin^2 60^0 = \frac{3}{4},$$

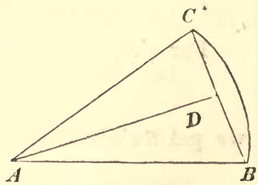
$$\text{or } \sin 60^0 = \frac{\sqrt{3}}{2} = \cos 30^0,$$

the positive sign being taken, since 60^0 is less than $\frac{\pi}{2}$.

The sine and cosine of 18^0 or $\frac{\pi}{10}$.

768. The sine of 18^0 is $\frac{\sqrt{5}-1}{4}$, and the cosine of 18^0 is $\frac{\sqrt{(10+2\sqrt{5})}}{4}$.

For if CB be the side of a decagon inscribed in a circle, its length is the greater of the two portions into which the radius AB is divided, when cut in extreme and mean ratio (Euclid, Book IV, Prop. 10): it follows therefore that



$$CB = \frac{\sqrt{5}-1}{2} \times AB \text{ (Art. 665, Ex. 3)},$$

and the angle CAB which it subtends is $\frac{360^0}{10}$ or 36^0.

If we draw AD perpendicular to CB, it bisects the angle

BAC and also the chord BC: the angle BAD is therefore 18^0, and the semichord $BD = \frac{\sqrt{5}-1}{4} \times AB$: and since the sine of

$$BAD = \frac{BD}{AB} = \frac{\sqrt{5}-1}{4},$$

it follows that $\sin 18^0 = \frac{\sqrt{5}-1}{4}$, and

$$\cos 18^0 = \sqrt{(1-\sin^2 18^0)} = \frac{\sqrt{(10+2\sqrt{5})}}{4} \quad \text{(Art. 758).}$$

769. It will readily follow from the three last Articles, and those which precede them, that

1. $\text{Sin } 135^0 = \frac{1}{\sqrt{2}}$.
2. $\text{Sin } 225^0 = \sin 315^0 = -\frac{1}{\sqrt{2}}$.
3. $\text{Cos } 135^0 = -\frac{1}{\sqrt{2}} = \cos 225^0$.
4. $\text{Cos } 315^0 = \frac{1}{\sqrt{2}}$.
5. $\text{Sin } 120^0 = \frac{1}{2}$.
6. $\text{Sin } 210^0 = \sin 330^0 = -\frac{1}{2}$.
7. $\text{Cos } 120^0 = \cos 210^0 = -\frac{\sqrt{3}}{2}$.
8. $\text{Cos } 330^0 = \frac{\sqrt{3}}{2}$.
9. $\text{Sin } 150^0 = \frac{\sqrt{3}}{2}$.
10. $\text{Sin } 240^0 = \sin 300^0 = -\frac{\sqrt{3}}{2}$.
11. $\text{Cos } 120^0 = \cos 240^0 = -\frac{1}{2}$.
12. $\text{Cos } 300^0 = \frac{1}{2}$.
13. $\text{Sin } 108^0 = \frac{\sqrt{(10+2\sqrt{5})}}{4}$.
14. $\text{Sin } 208^0 = \sin 342^0 = -\frac{\sqrt{5}-1}{4}$.
15. $\text{Cos } 108^0 = \cos 252^0 = -\frac{\sqrt{5}-1}{4}$.
16. $\text{Cos } 342^0 = \frac{\sqrt{(10+2\sqrt{5})}}{4}$.

The numerical values of the sines and cosines of any multiple and submultiple of an angle, whose sine and cosine is given, may be determined by the aid of the following proposition and

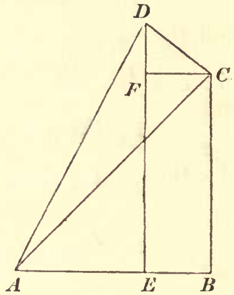
those which follow it, which are of fundamental importance in the theory of the sines and cosines of angles.

Given the sines and cosines of two angles, to find the sine and cosine of their sum and difference.

770. (Proposition.) Given the sines and cosines of two angles, to find the sine and cosine of their sum and difference.

Let the angles whose sines and cosines are given, be θ and θ', and let it be required to find the sines and cosines of $\theta + \theta'$ and $\theta - \theta'$.

Let the angle $BAC = \theta$, $CAD = \theta'$, and therefore $BAD = \theta + \theta'$: and let CB be drawn perpendicular to AB, CD to AD, DE to AB, and CF to DE.



Then, from Art. 757, we get

$$AC = AD \cos\theta' \text{ and } CD = AD \sin\theta':$$

and also

$$BC = AC \sin\theta \text{ and } AB = AC \cos\theta:$$

and if, in the expressions for BC and AB, we replace AC by $AD \cos\theta'$, we shall get

$$BC = AD \sin\theta \cos\theta' \text{ and } AB = AD \cos\theta \cos\theta'.$$

Again, $CF = CD \sin CDF = CD \sin\theta$ and $DF = CD \cos\theta$, since the angle CDF is equal to BAC or θ: and if, in the expressions for CF and DF, we replace CD by $AD \sin\theta'$, we shall get

$$CF = BE = AD \sin\theta \sin\theta' \text{ and } DF = AD \cos\theta \sin\theta',$$

and therefore

$$DE = BC + DF = AD \sin\theta \cos\theta' + AD \cos\theta \sin\theta',$$

and

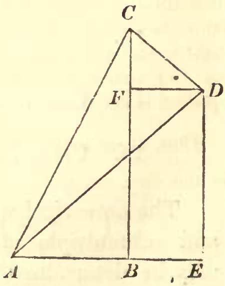
$$AE = AB - CF = AD \cos\theta \cos\theta' - AD \sin\theta \sin\theta'.$$

We thus find

$$\sin(\theta + \theta') = \frac{DE}{AD} = \sin\theta \cos\theta' + \cos\theta \sin\theta' \qquad (a),$$

$$\cos(\theta + \theta') = \frac{AE}{AD} = \cos\theta \cos\theta' - \sin\theta \sin\theta' \qquad (b).$$

Again, if we make $BAC = \theta$, $CAD = \theta'$, and therefore $BAD = \theta - \theta'$; and if we draw CB perpendicular to AB, CD to AC, DE to AB produced, and DF to CB, we shall find, as before,



$$AC = AD \cos\theta' \text{ and } CD = AD \sin\theta':$$

and also

$$BC = AC \sin\theta \text{ and } AB = AC \cos\theta:$$

and if, in the expressions for BC and AB, we replace AC by $AD\cos\theta'$, we shall get

$$BC = AD\sin\theta\cos\theta' \text{ and } AB = AD\cos\theta\cos\theta'.$$

Also $CF = CD\cos DCF = CD\cos\theta$ and $DF = CD\sin\theta$, since the angle DCF is equal to the angle BAC or θ: and if, in the expressions for CF and DF, we replace CD by $AD\sin\theta'$, we shall get

$$CF = AD\cos\theta\sin\theta' \text{ and } DF = AD\sin\theta\sin\theta',$$

and therefore

$$DE = BC - CF = AD\sin\theta\cos\theta' - AD\cos\theta\sin\theta',$$

and

$$AE = AB + DF = AD\cos\theta\cos\theta' + AD\sin\theta\sin\theta'.$$

We thus find

$$\sin(\theta - \theta') = \frac{DE}{AD} = \sin\theta\cos\theta' - \cos\theta\sin\theta' \qquad (c),$$

and

$$\cos(\theta - \theta') = \frac{AE}{AD} = \cos\theta\cos\theta' + \sin\theta\sin\theta' \qquad (d).$$

771. Of the four formulæ (a), (b), (c) and (d), which are given in the preceding Article, the three last may be easily derived from the first (a) or from

Three of the formulæ in the last Article may be derived from the first.

$$\sin(\theta + \theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta'^{*} \qquad (a),$$

by the aid of propositions previously established in this Chapter, provided we assume this formula to be true for all values of θ and θ'.

* The investigation of this formula, which is given in Art. 770, is incomplete, unless it is made to comprehend all values of θ, θ' and $\theta + \theta'$, which are within the limits of geometrical angles, to *which the definition of sine and cosine extends: for it is only when definitions cease to be applicable, that "the principle of the permanence of equivalent forms" comes into operation:* this might be effected geometrically, though not without some difficulty, by a proper adjustment of the figures to all the different cases which this proposition comprehends; a very slight examination, however, of the forms which the two members of the equation (a) assume for any values whatsoever of θ and θ', will shew that the equation is correct in all cases.

Thus, if we replace θ by $\frac{\pi}{2} + \theta$, and assume the formula (a) to be correct for this case, we get

$$\sin\left(\frac{\pi}{2} + \theta + \theta'\right) = \sin\left(\frac{\pi}{2} + \theta\right)\cos\theta' + \cos\left(\frac{\pi}{2} + \theta\right)\sin\theta':$$

and

Thus, if in (a) we replace θ' by $-\theta$, we shall find

$$\sin -\theta' = -\sin\theta' \text{ and } \cos -\theta' = \cos\theta',$$

and replacing $\sin\left(\frac{\pi}{2}+\theta+\theta'\right)$ by $\cos(\theta+\theta')$,

$$\sin\left(\frac{\pi}{2}+\theta\right) \text{ by } \cos\theta \text{ and } \cos\left(\frac{\pi}{2}+\theta\right) \text{ by } -\sin\theta, \text{ (Art. 764)},$$

we obtain

$$\cos(\theta+\theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \qquad (b),$$

an equation which has been shewn to be correct when θ, θ' and $\theta+\theta'$ are less than $\frac{\pi}{2}$: the formula (a) assumed, therefore, is correct in this case, since it leads to a correct conclusion.

Again, if we replace θ by $\frac{\pi}{2}+\theta$, θ' by $\frac{\pi}{2}+\theta'$, and assume the formula (a) to be correct for this case also, we shall get

$$\sin(\pi+\theta+\theta') = \sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}+\theta'\right) + \cos\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}+\theta\right),$$

and replacing $\sin(\pi+\theta+\theta')$ by $-\sin(\theta+\theta')$ (Art. 765), $\sin\left(\frac{\pi}{2}+\theta\right)$ by $\cos\theta$, $\cos(\pi+\theta')$ by $-\sin\theta'$, $\cos\left(\frac{\pi}{2}+\theta\right)$ by $-\sin\theta$, and $\sin\left(\frac{\pi}{2}+\theta'\right)$ by $\cos\theta'$, (Art. 764), we obtain

$$-\sin(\theta+\theta') = -\cos\theta\sin\theta' - \sin\theta\cos\theta':$$

or changing all the signs,

$$\sin(\theta+\theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta' \qquad (a),$$

a correct equation.

If, in the same formula (a) we further replace θ by $\pi+\theta$ and θ' by $\frac{\pi}{2}+\theta'$, and assume its correctness for this case also, we shall get

$$\sin\left(\frac{3\pi}{2}+\theta+\theta'\right) = \sin(\pi+\theta)\cos\left(\frac{\pi}{2}+\theta'\right) + \cos(\pi+\theta)\sin\left(\frac{\pi}{2}+\theta'\right):$$

if we now replace $\sin\left(\frac{3\pi}{2}+\theta+\theta'\right)$ by $-\cos(\theta+\theta')$ (Art. 765), $\sin(\pi+\theta)$ by $-\sin\theta$, $\cos(\pi+\theta)$ by $-\cos\theta$, $\sin\left(\frac{\pi}{2}+\theta'\right)$ by $\cos\theta'$, and $\cos\left(\frac{\pi}{2}+\theta'\right)$ by $-\sin\theta'$, we shall obtain

$$-\cos(\theta+\theta') = \sin\theta\sin\theta' - \cos\theta\cos\theta',$$

or, changing all the signs,

$$\cos(\theta+\theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \qquad (b),$$

a correct equation.

It is obvious that we may apply the same process to the verification of the fundamental formula $\sin(\theta+\theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta'$, for all values of θ, θ', and $\theta+\theta'$ whatsoever, including those in which θ or θ' become equal to $\frac{\pi}{2}$ or to any multiple of $\frac{\pi}{2}$, which would require a distinct geometrical investigation.

and therefore

$$\sin(\theta - \theta') = \sin\theta \cos\theta' - \sin\theta' \cos\theta \qquad (b).$$

Again, if in (b) we replace θ by $\frac{\pi}{2} - \theta$, we get

$$\sin\left(\frac{\pi}{2} - \theta - \theta'\right) = \sin\left(\frac{\pi}{2} - \theta\right)\cos\theta' - \cos\left(\frac{\pi}{2} - \theta\right)\sin\theta':$$

and if we further replace $\sin\left(\frac{\pi}{2} - \theta - \theta'\right)$ by $\cos(\theta + \theta')$,

$$\sin\left(\frac{\pi}{2} - \theta\right) \text{ by } \cos\theta \text{ and } \cos\left(\frac{\pi}{2} - \theta\right) \text{ by } \sin\theta,$$

we shall get

$$\cos(\theta + \theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \qquad (c).$$

If, in this formula (c) we replace θ' by $-\theta'$, we get

$$\cos(\theta - \theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta' \qquad (d),$$

observing that $\cos - \theta'$ is replaced by $\cos\theta'$, and $\sin - \theta'$ by $-\sin\theta'$.

772. Inasmuch as

Other derivative formulæ.

$$\sin(\theta + \theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta' \qquad (a),$$

$$\sin(\theta + \theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta' \qquad (b),$$

we readily obtain by the addition of (a) to (b),

$$\sin(\theta + \theta') + \sin(\theta - \theta') = 2\sin\theta\cos\theta' \qquad (e),$$

and by the subtraction of (b) from (a),

$$\sin(\theta + \theta') - \sin(\theta - \theta') = 2\cos\theta\sin\theta' \qquad (f).$$

And again, since

$$\cos(\theta + \theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \qquad (c),$$

$$\cos(\theta - \theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta' \qquad (d),$$

we find, by the addition of (d) to (c),

Formulæ for expressing the sum or difference of the sines or cosines of two angles in terms of the product of the sines or cosines of their semi-sum and semi-difference.

$$\cos(\theta + \theta') + \cos(\theta - \theta') = 2\cos\theta\cos\theta' \qquad (g),$$

and by the subtraction of (d) from (c),

$$\cos(\theta + \theta') - \cos(\theta - \theta') = -2\sin\theta\sin\theta' \qquad (h).$$

773. Inasmuch as it may be readily shewn that

$$\theta = \left(\frac{\theta + \theta'}{2}\right) + \left(\frac{\theta - \theta'}{2}\right) = s + d,$$

$$\theta' = \left(\frac{\theta + \theta'}{2}\right) - \left(\frac{\theta - \theta'}{2}\right) = s - d,$$

we get

$$\sin\theta + \sin\theta' = \sin(s+d) + \sin(s-d) = 2\sin s\cos d,$$

$$\sin\theta - \sin\theta' = \sin(s+d) - \sin(s-d) = 2\cos s\sin d,$$

$$\cos\theta + \cos\theta' = \cos(s+d) + \cos(s-d) = 2\cos s\cos d,$$

$$\cos\theta - \cos\theta' = \cos(s+d) - \cos(s-d) = -2\sin s\sin d:$$

and if we replace s by $\dfrac{\theta+\theta'}{2}$ and d by $\dfrac{\theta-\theta'}{2}$, we find

$$\sin\theta + \sin\theta' = 2\sin\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right) \qquad (i),$$

$$\sin\theta - \sin\theta' = 2\cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right) \qquad (k),$$

$$\cos\theta + \cos\theta' = 2\cos\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right) \qquad (l),$$

$$\cos\theta - \cos\theta' = -2\sin\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right) \qquad (m),$$

These formulæ, expressing the sum and difference of the sines and also of the cosines of two angles, in terms of the products of the sines and cosines of half their sum and half their difference, are very frequently referred to, and will be found to be extremely useful in the transformation of formulæ to others which are equivalent to them, and more particularly in their adaptation to logarithmic computation.

774. If we resume the formulæ

$$\sin(\theta+\theta') + \sin(\theta-\theta') = 2\sin\theta\cos\theta' \qquad (e),$$

$$\cos(\theta+\theta') + \cos(\theta-\theta') = 2\cos\theta\cos\theta' \qquad (g),$$

and if we transpose $\sin(\theta-\theta')$ in one case, and $\cos(\theta-\theta')$ in the other, from the left-hand side of the equation to the right, they will assume the form

$$\sin(\theta+\theta') = 2\sin\theta\cos\theta' - \sin(\theta-\theta') \qquad (n),$$

$$\cos(\theta+\theta') = 2\cos\theta\cos\theta' - \cos(\theta-\theta') \qquad (o).$$

If we further replace θ by $(n-1)\phi$ and θ' by ϕ, we get $\theta+\theta' = n\phi$, and $\theta-\theta' = (n-2)\phi$, and the formulæ (n) and (o), become

$$\sin n\phi = 2\cos\phi\sin(n-1)\phi - \sin(n-2)\phi \quad (p),$$

$$\cos n\phi = 2\cos\phi\cos(n-1)\phi - \cos(n-2)\phi \quad (q).$$

These formulæ are useful, as expressing the sines and cosines of multiples of an angle in terms of sines and cosines of inferior multiples of the same angle: they are very frequently referred to.

775. If in the formulæ given in the last Article, we replace n by 2, we get

To express the sine and cosine of double an angle in terms of the sine or cosine of the angle.

$$\sin 2\phi = 2 \sin \phi \cos \phi - \sin 0$$
$$= 2 \sin \phi \cos \phi:$$

for $\sin 0 = 0$:

$$\cos 2\phi = 2 \cos \phi \cos \phi - \cos 0$$
$$= 2 \cos^2 \phi - 1 = 1 - 2 \sin^2 \phi:$$

for $\cos 0 = 1$.

It follows, therefore, that if the sine and cosine of an angle be given, we can find by these formulæ the sine of double the angle: and also, if *either* the sine or cosine of an angle be given, we can find the cosine of double the angle.

Thus, the sine of 30^0 is $\frac{1}{2}$, and its cosine $\frac{\sqrt{3}}{2}$: therefore

$$\sin 60^0 = 2 \sin 30 \cos 30$$
$$= 2 \times \tfrac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

The sine of 18^0 is $\frac{\sqrt{5}-1}{4}$, and its cosine $\frac{\sqrt{10+2\sqrt{5}}}{4}$: therefore

$$\sin 36^0 = 2 \sin 18^0 \cos 18^0$$
$$= \frac{\sqrt{5}-1}{2} \times \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}.$$

Similarly, from the second formula, we get

$$\cos 36^0 = 2 \cos^2 18^0 - 1 = 1 - 2 \sin^2 18^0$$
$$= \frac{\sqrt{5}+1}{4}.$$

776. The same formulæ will likewise enable us to express the sine or cosine of half an angle in terms of the sine or cosine of the angle itself.

Given the sine or cosine of an angle, to find the sine and cosine of half the angle.

Thus, let it be required to express the values of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in terms of $\sin\theta$.

Inasmuch as

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

and

$$1 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} \quad \text{(Art. 758)}$$

we get, by adding

$$1 + \sin\theta = \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2},$$

and, by subtracting

$$1 - \sin\theta = \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}.$$

If $\sin\frac{\theta}{2}$ be greater than $\cos\frac{\theta}{2}$, the terms of the complete squares on the right-hand side of these equations are arranged in the order of their magnitude (Arts. 587 and 650), and the extraction of their square roots will give us

$$\sqrt{(1 + \sin\theta)} = \sin\frac{\theta}{2} + \cos\frac{\theta}{2}$$

$$\sqrt{(1 - \sin\theta)} = \sin\frac{\theta}{2} - \cos\frac{\theta}{2};$$

and therefore, by adding and subtracting, we get

$$\sin\frac{\theta}{2} = \tfrac{1}{2}\sqrt{(1 + \sin\theta)} + \tfrac{1}{2}\sqrt{(1 - \sin\theta)} \qquad (1),$$

$$\cos\frac{\theta}{2} = \tfrac{1}{2}\sqrt{(1 + \sin\theta)} - \tfrac{1}{2}\sqrt{(1 - \sin\theta)} \qquad (2).$$

But, if $\sin\frac{\theta}{2}$ be less than $\cos\frac{\theta}{2}$, the order of the terms in the squares

$$\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}, \text{ and } \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2},$$

must be reversed: the extraction of their roots, when their form is thus modified, will in that case give us

$$\sqrt{(1+\sin\theta)} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}$$

$$\sqrt{(1-\sin\theta)} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2};$$

and therefore

$$\cos\frac{\theta}{2} = \tfrac{1}{2}\sqrt{(1+\sin\theta)} + \tfrac{1}{2}\sqrt{(1-\sin\theta)} \qquad (3),$$

$$\sin\frac{\theta}{2} = \tfrac{1}{2}\sqrt{(1+\sin\theta)} - \tfrac{1}{2}\sqrt{(1-\sin\theta)} \qquad (4).$$

If θ be between the limits 0 and $\frac{\pi}{2}$, $\frac{3\pi}{2}$ and 2π, we must use the expressions (3) and (4): but if θ be between the limits $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, we must use the expressions (1) and (2): it should be kept in mind, however, that for greater values of θ, which make $\frac{\theta}{2}$ correspond to a negative geometrical angle, these expressions must change their sign. (See Art. 781).

Again, let it be required to express $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in terms of $\cos\theta$.

Since

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2},$$

we readily obtain the equations

$$2\cos^2\frac{\theta}{2} = 1 + \cos\theta,$$

$$2\sin^2\frac{\theta}{2} = 1 - \cos\theta,$$

and therefore

$$\cos\frac{\theta}{2} = \sqrt{\left(\frac{1+\cos\theta}{2}\right)} \qquad (5),$$

$$\sin\frac{\theta}{2} = \sqrt{\left(\frac{1-\cos\theta}{2}\right)} \qquad (6).$$

If θ be between the limits π and 3π, 5π and 7π, 9π and 11π, it will be found that $\cos\frac{\theta}{2} = -\sqrt{\left(\frac{1-\cos\theta}{2}\right)}$: and if θ be between

the limits 2π and 4π, 6π and 8π, 10π and 12π, then $\sin\frac{\theta}{2}$ $= -\sqrt{\left(\frac{1-\cos\theta}{2}\right)}$.

Examples. 777. The following are examples.

(1) Given the sine of 30^0, which is $\frac{1}{2}$ (Art. 767), to find the sine and cosine of 15^0.

$$\sin 15^0 = \tfrac{1}{2}\sqrt{(1+\sin 30^0)} - \tfrac{1}{2}\sqrt{(1-\sin 30^0)}$$

$$= \tfrac{1}{2}\sqrt{\tfrac{3}{2}} - \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = .25899,$$

$$\cos 15^0 = \tfrac{1}{2}\sqrt{(1+\sin 30^0)} + \tfrac{1}{2}\sqrt{(1-\sin 30^0)}$$

$$= \tfrac{1}{2}\sqrt{\tfrac{3}{2}} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} = .96609.$$

(2) Given the cosine of $18^0 = \sqrt{\left(\frac{5+\sqrt{5}}{8}\right)} = .897$, to find the sine and cosine of 9^0.

$$\sin 9^0 = \sqrt{\left(\frac{1-\cos 18^0}{2}\right)} = \sqrt{(.0515)} = .2269,$$

$$\cos 9^0 = \sqrt{\left(\frac{1+\cos 18^0}{2}\right)} = \sqrt{(.973)} = .9729.$$

It is obvious that if the sine or cosine of any angle θ be given, we can apply the same method to find the sine and cosine of $\frac{\theta}{2^n}$, whatever be the value of n.

To determine a series of equisinal angles. 778. To find the series of values of θ which have a given sine*.

In the first place, all angles are *equisinal* or have the same sine, whose measures differ from each other by multiples of 2π. (Art. 761).

In the second place, if θ be an angle whose sine is of the required magnitude, its supplement $\pi - \theta$ is also equisinal with it. (Art. 763).

* Or, in other words, to find all the values of θ, which satisfy the equation $\sin\theta = a$, where a is any positive or negative number less than 1.

It will follow, therefore, that all the angles in the two series

$$\theta, \quad 2\pi+\theta, \quad 4\pi+\theta, \ldots \quad 2n\pi+\theta, \ldots$$
$$\pi-\theta, \quad 3\pi-\theta, \quad 5\pi-\theta, \ldots \quad (2n+1)\pi-\theta,$$

continued both ways, are equisinal with θ: or in other words

$$\sin\theta = \sin(2n\pi+\theta) = \sin\{(2n+1)\pi-\theta\}$$

when n is zero or any whole number, whether positive or negative.

779. To find the series of values of θ which have a given cosine. To determine a series of equicosinal angles.

In the first place, all angles are *equicosinal* or have the same cosine, whose measures differ from each other by multiples of 2π. (Art. 761).

In the second place, if θ be an angle whose cosine is of the required magnitude, then $-\theta$ is *equicosinal* with it. (Art. 762).

It will follow, therefore, that all the angles in the two series

$$\theta, \quad 2\pi+\theta, \quad 4\pi+\theta, \ldots \quad 2n\pi+\theta, \ldots$$
$$-\theta, \quad 2\pi-\theta, \quad 4\pi-\theta, \ldots \quad 2n\pi-\theta, \ldots$$

continued both ways, are equicosinal with θ: or, in other words,

$$\cos\theta = \cos(2n\pi \pm \theta),$$

where n is zero or any positive or negative whole number.

780. The propositions in the two last Articles deserve the most careful consideration of the student, as furnishing the basis of the explanation of the multiple values not merely of equisinal and equicosinal angles, but likewise of the periods formed by the sines and cosines of their submultiples, which will be found to be intimately connected with some of the most important theories in analysis. Great importance of the theory of the sines and cosines of equisinal and equicosinal angles, and of their submultiples.

Thus, if the angle be given, there is only one value of its sine and cosine: there is also only one value of the sine or cosine of any given multiple or submultiple of it, which form likewise a unique and determinate angle.

If however the sine or cosine of an angle be given, and *the value of the angle be required to be determined from the value of its sine or cosine,* then not only may the angle corresponding be any one or more of the series of *equisinal* or *equicosinal*

angles, but the sines and cosines of their submultiples will be periodic.

The periods of the sines of the submultiples of equisinal angles.

781. Thus the angles in the series $2n\pi+\theta$ are equisinal with θ: but the sines of a series of angles, which are $\frac{1}{m}^{\text{th}}$ part of those in the series $2n\pi+\theta$, form periodical series, the sines of the first m terms being different from each other, and afterwards recurring periodically in the same order.

Thus, the equisinal series being

$$\theta, \quad 2\pi+\theta, \quad 4\pi+\theta, \ldots \quad 2(m-1)\pi+\theta, \quad 2m\pi+\theta, \ldots$$

the series of their submultiples by m, will be

$$\frac{\theta}{m}, \quad \frac{2\pi+\theta}{m}, \quad \frac{4\pi+\theta}{m}, \ldots \quad \frac{2(m-1)\pi+\theta}{m}, \quad 2\pi+\frac{\theta}{m},$$

and the sines of the m first terms are different from each other: but the $(m+1)^{\text{th}}$ term is $2\pi+\frac{\theta}{m}$ which is equisinal with $\frac{\theta}{m}$, the $(m+2)^{\text{th}}$ term is $2\pi+\frac{2\pi+\theta}{m}$, which is equisinal with $\frac{2\pi+\theta}{m}$, and so on successively: it follows therefore that the series of sines of the series of submultiple angles is periodic, each period consisting of m terms.

Again, if we take the second series of equisinal angles, included in the formula $(2n+1)\pi-\theta$, which is

$$\pi-\theta, \quad 3\pi-\theta, \quad 5\pi-\theta, \ldots \quad (2m-1)\pi-\theta, \quad (2m+1)\pi-\theta, \ldots$$

the corresponding series of their submultiples by m will be

$$\frac{\pi-\theta}{m}, \quad \frac{3\pi-\theta}{m}, \quad \frac{5\pi-\theta}{m}, \ldots \quad \frac{(2m-1)\pi-\theta}{m}, \quad 2\pi+\frac{\pi-\theta}{m}, \ldots$$

and the sines of the m first terms are different from each other; that of the $(m+1)^{\text{th}}$ term being equisinal with the first, of the $(m+2)^{\text{th}}$ with the second, and so on for ever. The sines therefore of the second series of submultiple angles or measures of angles is also periodic, each period consisting of m terms.

It follows therefore that there are, in the two periods, $2m$ submultiple angles, *less than* 360°, whose sines *may* be different from each other, corresponding to the same sine, or to the sine of any term of the two series of equisinal angles: but this number, when m is odd, may be reduced one half, by the identity

of the sines of the terms of one period with those of the other, the terms reckoned from the beginning of one period being severally the supplements of the terms of the other reckoned from the end, and therefore the second period may be altogether left out of consideration: and in the case in which m is even, the first half of the terms of each period will bear to those of the second the relation of ϕ and $\pi+\phi$, whose sines are equal, but with different signs.

782. If the cosines of the angles which are the $\frac{1}{m}^{\text{th}}$ part of the equicosinal angles included in the formula $2n\pi\pm\theta$ (Art. 779) be required, we form the double series The periods of the cosines of the submultiples of equicosinal angles.

$$\pm\theta,\ 2\pi\pm\theta,\ 4\pi\pm\theta, \dots\ 2(m-1)\pi+\theta,\ 2m\pi\pm\theta, \dots$$

and the cosines of the first m terms of the series of submultiples

$$\pm\frac{\theta}{m},\quad \frac{2\pi}{m}\pm\frac{\theta}{m},\quad \frac{4\pi}{m}\pm\frac{\theta}{m},\dots\quad \frac{2(m-1)\pi}{m}\pm\frac{\theta}{m},\quad 2\pi\pm\frac{\theta}{m},$$

will be different from each other, comprising $2m$ different angles: but whether m be odd or even, we find $\cos\frac{\theta}{m}=\cos-\frac{\theta}{m}$, $\cos\left(\frac{2\pi}{m}\pm\frac{\theta}{m}\right)=\cos\left\{\frac{(2m-1)\pi}{m}\mp\frac{\theta}{m}\right\}$, $\cos\left(\frac{4\pi}{m}\pm\frac{\theta}{m}\right)=\cos\left\{\frac{2(m-2)\pi}{m}\mp\frac{\theta}{m}\right\}$, thus reducing the number of different values of the cosine from $2m$ to m: and further when m is even, the first $\frac{m}{2}$ terms from the beginning of the period are the supplements of the last $\frac{m}{2}$ terms reckoned from the $\left(\frac{m}{2}+1\right)^{\text{th}}$ term to the end, and their several cosines therefore differ from each other in their signs only.

783. The following are examples: Examples.

(1) Given $\sin\theta$, to find $\sin\frac{\theta}{2}$.

The series of equisinal angles are

$$\theta,\ 2\pi+\theta,\ 4\pi+\theta, \dots\dots$$

$$\pi-\theta,\ 3\pi-\theta,\ 5\pi-\theta, \dots\dots$$

The corresponding series of semi-angles are

$$\frac{\theta}{2},\ \pi+\frac{\theta}{2},\ 2\pi+\frac{\theta}{2},\ \ldots\ldots$$

$$\frac{\pi}{2}-\frac{\theta}{2},\ \frac{3\pi}{2}-\theta,\ \frac{5\pi}{2}-\frac{\theta}{2},\ \ldots\ldots$$

The periods of sines are

$$\sin\frac{\theta}{2},\ \sin\left(\pi+\frac{\theta}{2}\right),\text{ and}$$

$$\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right),\ \sin\left(\frac{3\pi}{2}-\frac{\theta}{2}\right),$$

of which $\sin\frac{\theta}{2}=-\sin\left(\pi+\frac{\theta}{2}\right)$ and

$$\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=-\sin\left(\frac{3\pi}{2}-\frac{\theta}{2}\right)=\cos\frac{\theta}{2}.$$

The expression, given in Art. 776, if we assign the double sign to the square roots, or

$$\pm\tfrac{1}{2}\sqrt{\left(\frac{1+\sin\theta}{2}\right)}\pm\tfrac{1}{2}\sqrt{\left(\frac{1-\sin\theta}{2}\right)},$$

will express these four values, two of them being equal to $\pm\sin\frac{\theta}{2}$, and the others to $\pm\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$ or $\pm\cos\frac{\theta}{2}$.

(2) Given $\cos\theta$, to find $\cos\frac{\theta}{2}$.

The double series of equicosinal angles is

$$\pm\theta,\ 2\pi\pm\theta,\ 4\pi\pm\theta,\ \ldots\ldots$$

The corresponding series of semi-angles is

$$\pm\frac{\theta}{2},\ \pi\pm\frac{\theta}{2},\ 2\pi\pm\frac{\theta}{2},\ \ldots\ldots$$

The period of cosines is

$$\cos\pm\frac{\theta}{2},\ \cos\left(\pi\pm\frac{\theta}{2}\right),$$

of which $\cos\frac{\theta}{2}=\cos-\frac{\theta}{2}$ and $\cos\left(\pi\pm\frac{\theta}{2}\right)=-\cos\frac{\theta}{2}$.

The expression, given in Art. 776,

$$\cos\frac{\theta}{2}=\pm\sqrt{\left(\frac{1+\cos\theta}{2}\right)}$$

will express both these values, if we assign to it its double sign.

(3) Given $\sin\theta$, to find $\sin\frac{\theta}{3}$.

The two series of equisinal angles are

$$\theta,\ 2\pi+\theta,\ 4\pi+\theta,\ 6\pi+\theta, \ldots\ldots$$
$$\pi-\theta,\ 3\pi-\theta,\ 5\pi-\theta,\ 7\pi-\theta, \ldots\ldots$$

The corresponding series of ternary submultiples of these angles are

$$\frac{\theta}{3},\ \frac{2\pi}{3}+\frac{\theta}{3},\ \frac{4\pi}{3}+\frac{\theta}{3},\ 2\pi+\frac{\theta}{3}, \ldots\ldots$$
$$\frac{\pi}{3}-\frac{\theta}{3},\ \frac{3\pi}{3}-\frac{\theta}{3},\ \frac{5\pi}{3}-\frac{\theta}{3},\ \frac{7\pi}{3}-\frac{\theta}{3}, \ldots\ldots$$

The periods of sines are

$$\sin\frac{\theta}{3},\ \sin\left(\frac{2\pi}{3}+\frac{\theta}{3}\right),\ \sin\left(\frac{4\pi}{3}+\frac{\theta}{3}\right),$$

and

$$\sin\left(\frac{\pi}{3}-\frac{\theta}{3}\right),\ \sin\left(\frac{3\pi}{3}-\frac{\theta}{3}\right),\ \sin\left(\frac{5\pi}{3}-\frac{\theta}{3}\right);$$

but since $\frac{\theta}{3}$ is the supplement of $\frac{3\pi}{3}-\frac{\theta}{3}$, $\frac{2\pi}{3}+\frac{\theta}{3}$ of $\frac{\pi}{3}-\frac{\theta}{3}$, and $\frac{4\pi}{3}+\frac{\theta}{3}$ of $\frac{5\pi}{3}-\frac{\theta}{3}$ (after rejecting 2π), it follows that the values of the sines of the first period are identical with those of the second: and there are consequently only three such values, which are different from each other*.

(4) Given $\cos\theta$, to find $\cos\frac{\theta}{4}$.

The double series of equicosinal angles is

$$\pm\theta,\ 2\pi\pm\theta,\ 4\pi\pm\theta,\ 6\pi\pm\theta,\ 8\pi\pm\theta.\ldots\ldots$$

The corresponding double series of quaternary submultiples of the angles of this series is

$$\pm\frac{\theta}{4},\ \frac{\pi}{2}\pm\frac{\theta}{4},\ \pi\pm\frac{\theta}{4},\ \frac{3\pi}{2}\pm\frac{\theta}{4},\ 2\pi\pm\frac{\theta}{4}.$$

* It appears by the formula in Art. 774, that

$$\sin\theta = 2\cos\frac{\theta}{3}\sin\frac{2\theta}{3}-\sin\frac{\theta}{3}$$
$$= 4\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}-\sin\frac{\theta}{3}\ \left(\text{replacing } \sin\frac{2\theta}{3} \text{ by } 2\cos\frac{\theta}{3}\sin\frac{\theta}{3}\right)$$
$$= 4\left(1-\sin^2\frac{\theta}{3}\right)\sin\frac{\theta}{3}-\sin\frac{\theta}{3}$$
$$= 3\sin\frac{\theta}{3}-4\sin^3\frac{\theta}{3},$$

a cubic equation, whose roots are $\sin\frac{\theta}{3}$, $\sin\left(\frac{\pi}{3}-\frac{\theta}{3}\right)$, and $-\sin\left(\frac{2\pi}{3}-\frac{\theta}{3}\right)$, where θ is the least angle whose sine is of the given magnitude.

The double period of cosines is

$$\cos \pm \frac{\theta}{4}, \quad \cos\left(\frac{\pi}{2} \pm \frac{\theta}{4}\right), \quad \cos\left(\pi \pm \frac{\theta}{4}\right), \quad \cos\left(\frac{3\pi}{2} \pm \frac{\theta}{4}\right),$$

which are reducible to

$$\cos \frac{\theta}{4}, \quad \cos\left(\frac{\pi}{2} - \frac{\theta}{4}\right), \quad \cos\left(\frac{\pi}{2} + \frac{\theta}{4}\right), \quad \cos\left(\pi - \frac{\theta}{4}\right),$$

which alone are different from each other, and of which the fourth and third only differ from the first and second in their sign *.

Caution respecting the application of the principle of equivalent forms.

784. It should be observed, that the definitions of the sine and cosine apply to all geometrical angles, and therefore to those which are negative as well as positive, extending thus beyond the limits of Arithmetical Algebra: and whatever be the propositions, which come within the range of these definitions, they must be established by means of them, and not by the generalizations of Symbolical Algebra: and though it is by the aid of Symbolical Algebra alone that we are enabled to form and express negative lines, negative arcs, and negative angles, yet as soon as we have given to them a consistent interpretation, they come within the proper province of Geometry, and are subject to the same definitions and propositions, as apply to the geometrical quantities which they represent in affection as well as in magnitude: it is for this reason that in demonstrating the formulæ of Goniometry, we must consider the negative values of the quantities involved as equally possible with those which are positive, and as equally within the range of our definitions: thus, we are not at liberty to pass from the formula for $\sin(\theta + \theta')$ to that for $\sin(\theta - \theta')$, by the aid of the principle of the "permanence of equivalent forms," but by reasonings founded upon the definitions of the sine and cosine, and the geometrical or other properties of the magnitudes to which they apply: it is only when the definitions, which are the foundation of our reasonings cease to be applicable, that we are at liberty to resort to the generalization of Symbolical Algebra.

* The corresponding biquadratic equation is $\cos\theta = 8\cos^4 \frac{\theta}{4} - 8\cos^2 \frac{\theta}{4} + 1$.

CHAPTER XXVIII.

ON THE TANGENTS, COTANGENTS, SECANTS, COSECANTS, AND VERSED SINES OF ANGLES.

Definition of tangent and cotangent.

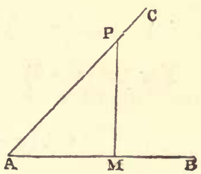
785. The ratio of the sine to the cosine of an angle is called its *tangent:* and the reciprocal ratio of the cosine to the sine of an angle is called its *cotangent**.

Thus, if θ be the angle, then the tangent of θ or $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

and the cotangent of θ or $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

They determine the angle.

786. The tangent and cotangent of an angle *determine* the angle equally with the sine and cosine: for if BAC be any angle, and if any point whatsoever P be taken in AC, and if PM be drawn perpendicular to AB, then the ratio $\frac{PM}{AM}$ is the *tangent*, and the reciprocal ratio $\frac{AM}{PM}$ is the *cotangent* of the angle BAC: and if this ratio be given, it determines the *species* of the triangle PAM, and therefore the angle at A, and conversely. Euclid, Book VI, Prop. 5.

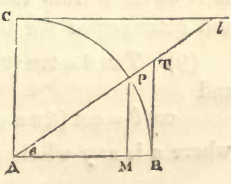


Properties of the tangent and cotangent.

787. The following properties of the tangent and cotangent are derivable immediately from their definitions.

(1) $\text{Tan}\,\theta = \frac{1}{\cot\theta}$.

* The tangent and cotangent were formerly defined as lines, and not as ratios: thus, if BP be the arc of a circle subtending the angle θ at the centre, and if BT and Ct be drawn from B and from the extremity C of the quadrant, tangents to the circle meeting AP produced in T and t respectively, then BT was called the *tangent*, and Ct the *cotangent* of the angle BAP or θ.



For $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$, and the first ratio is the reciprocal of the second, and conversely.

(2) $\operatorname{Tan} - \theta = - \tan\theta$, and $\cot - \theta = - \cot\theta$.

For $\tan - \theta = \frac{\sin - \theta}{\cos - \theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$, and

$$\cot - \theta = \frac{\cos - \theta}{\sin - \theta} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

(3) $\operatorname{Tan}\left(\frac{\pi}{2} - \theta\right) = \cot\theta$, and conversely.

For $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$:

$$\text{and } \cot\left(\frac{\pi}{2} - \theta\right) = \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

(4) $\operatorname{Tan}(\pi - \theta) = -\tan\theta$ and $\cot(\pi - \theta) = -\cot\theta$.

For $\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$: and

$$\cot(\pi - \theta) \frac{\cos(\pi - \theta)}{\sin(\pi - \theta)} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta.$$

By a similar process it may be shewn that

(5) $\operatorname{Tan}\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$ and $\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$.

(6) $\operatorname{Tan}\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$ and $\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$.

(7) $\operatorname{Tan}\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$ and $\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta$.

(8) $\operatorname{Tan}(\pi + \theta) = \tan\theta$ and $\cot(\pi + \theta) = \cot\theta$.

Inasmuch as the sines and cosines of all angles which differ by multiples of 2π or 360°, are identical, the same observation, as is obvious from their definitions, will extend to their tangents and cotangents: we thus get

(9) $\operatorname{Tan}\theta = \tan(2n\pi + \theta) = \tan(\pi + \theta) = \tan\{(2n+1)\pi + \theta\}$,
and

$$\cot\theta = \cot(2n\pi + \theta) = \cot(\pi + \theta) = \cot\{(2n+1)\pi + \theta\},$$

where n is any whole number, whether positive or negative.

(10) $\sqrt{(1+\tan^2\theta)} = \frac{1}{\cos\theta}$ and $\sqrt{(1+\cot^2\theta)} = \frac{1}{\sin\theta}$ *.

For $1+\tan^2\theta = 1+\frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$; and therefore $\sqrt{(1+\tan^2\theta)} = \frac{1}{\cos\theta}$: and again,

$$1+\cot^2\theta = 1+\frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta};$$

and therefore $\sqrt{(1+\cot^2\theta)} = \frac{1}{\sin\theta}$.

Reciprocally, we find

$$\cos\theta = \frac{1}{\sqrt{(1+\tan^2\theta)}} \text{ and } \sin\theta = \frac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}.$$

(11) $\operatorname{Tan}\theta + \cot\theta = \frac{2}{\sin 2\theta}$.

For $\tan\theta+\cot\theta = \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\frac{1}{2}\sin 2\theta}$

$$= \frac{2}{\sin 2\theta} \dagger.$$

(12) $\operatorname{Tan}\theta - \cot\theta = 2\cot 2\theta$.

For $\tan\theta-\cot\theta = \frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta-\cos^2\theta}{\cos\theta\sin\theta} = \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = 2\cot 2\theta$.

788. The tangent and cotangent may have every possible value between *zero* and *infinity*, whether positive or negative. Changes of sign and value of the tangent and cotangent.

For $\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$: and $\tan\frac{\pi}{2} = \frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0} = \infty$, or infinity: and for values of θ between 0 and $\frac{\pi}{2}$, the tangent increases with the increase of θ: for values of θ which are greater than $\frac{\pi}{2}$, and less than π, the value of the tangent is negative, and diminishes from infinity to zero, which it reaches when $\theta=\pi$:

* It will be seen in Art. 793, that $\frac{1}{\cos\theta}$ is called the secant of θ, and $\frac{1}{\sin\theta}$ the cosecant of θ.

† $\frac{2}{\sin 2\theta} = 2\operatorname{cosec} 2\theta$ (Art. 793), and the proposition in the text may be put under the following form, $\tan\theta+\cot\theta = 2\operatorname{cosec} 2\theta$.

for values of θ which are greater than π and less than $\frac{3\pi}{2}$, the tangent is *positive,* and increases from zero to infinity: and for values of θ between $\frac{3\pi}{2}$ and 2π, the tangent is negative, and decreases from infinity to zero.

The cotangent being the reciprocal of the tangent, its signs are the same for the same angle, but its values follow a reversed order, one being infinite when the other is *zero,* and one increasing when the other is diminishing.

It may be observed that the tangent and cotangent change their signs when they pass through *infinity* and *zero,* or, in other words, at the points where θ is 0, or $\frac{\pi}{2}$, or a multiple of $\frac{\pi}{2}$: this subject, which is not without importance, will be further considered in a subsequent Chapter*.

Given the tangents of two angles to find the tangent of their sum and difference.

789. PROPOSITION. Given the tangents of two angles, to find the tangent of their sum and difference.

Let θ and θ' be the angles, whose tangents are given, and let it be required to express the tangent of $\theta \pm \theta'$.

$$\text{Tan}\,(\theta + \theta') = \frac{\sin(\theta \pm \theta')}{\cos(\theta \pm \theta')} = \frac{\sin\theta \cos\theta' \pm \cos\theta \sin\theta'}{\cos\theta \cos\theta \mp \sin\theta \sin\theta'}:$$

(dividing the numerator and denominator by $\cos\theta \cos\theta'$),

$$= \frac{\dfrac{\sin\theta}{\cos\theta} \pm \dfrac{\sin\theta'}{\cos\theta'}}{1 \mp \dfrac{\sin\theta}{\cos\theta} \times \dfrac{\sin\theta'}{\cos\theta'}} = \frac{\tan\theta \pm \tan\theta'}{1 \mp \tan\theta \tan\theta'}.$$

In a similar manner it may be shewn that

$$\cot(\theta \pm \theta') = \frac{\cot\theta \cot\theta' \mp 1}{\cot\theta' \pm \cot\theta}.$$

Expressions for $\tan n\theta$ in terms of $\tan\theta$ and $\tan(n-1)\theta$.

790. If, in the expression for $\tan(\theta + \theta')$ we replace θ' by θ, and θ by $(n-1)\theta$, we shall get

$$\tan n\theta = \frac{\tan(n-1)\theta + \tan\theta}{1 - \tan\theta \tan(n-1)\theta}.$$

* The signs of geometrical angles change in passing through zero and two right angles: the signs of the sines and cosines change in passing through zero only.

791. If $n=2$, the formula in the last Article gives us

Expression for $\tan 2\theta$ in terms of $\tan\theta$, and its converse.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

If we solve this equation with respect to $\tan\theta$, we get

$$\tan\theta = -\cot 2\theta \pm \surd(1+\cot^2 2\theta).$$

If the value of 2θ is required to be determined from the value of $\tan 2\theta$, then the angle may be any term in either of the two series, (Art. 787, No. 9),

$$2\theta,\ 2\pi+2\theta,\ 4\pi+2\theta, \dots\dots$$

$$\pi+2\theta,\ 3\pi+2\theta,\ 5\pi+2\theta, \dots\dots$$

The corresponding series of semiangles are

$$\theta,\ \pi+\theta,\ 2\pi+\theta, \dots\dots$$

$$\frac{\pi}{2}+\theta,\ \frac{3\pi}{2}+\theta,\ 2\pi+\frac{\pi}{2}+\theta, \dots\dots$$

and the tangents of all the angles in the first series are identical with $\tan\theta$, and in the second series with $-\cot\theta$: and it will be found that

$$\tan\theta = -\cot 2\theta + \surd(1+\cot^2 2\theta)^*$$

$$-\cot\theta = -\cot 2\theta - \surd(1+\cot^2 2\theta).$$

792. The following formulæ are not unfrequently useful, and are very easily proved:

$$(1)\quad \tan\theta + \tan\theta' = \frac{\sin(\theta+\theta')}{\cos\theta\cos\theta'}.$$

$$(2)\quad \tan\theta - \tan\theta' = \frac{\sin(\theta-\theta')}{\cos\theta\cos\theta'}.$$

* Since $\surd(1+\cot^2 2\theta) = \operatorname{cosec} 2\theta$ (Art. 795), it follows that

$$\tan\theta = \operatorname{cosec} 2\theta - \cot 2\theta,$$

$$\cot\theta = \operatorname{cosec} 2\theta + \cot 2\theta.$$

If in the formula in Art. 790, we make $n=3$, we get

$$\tan 3\theta = \frac{\tan 2\theta + \tan\theta}{1-\tan\theta\tan 2\theta} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta},$$

replacing $\tan 2\theta$ by $\dfrac{2\tan\theta}{1-\tan^2\theta}$.

The three roots of this equation are found by a process similar to that given in the text, and are $\tan\theta$, $\tan(60^\circ+\theta)$, $\tan(120^\circ+\theta)$, or

$$\tan\theta,\ \frac{\surd 3+\tan\theta}{1-\surd 3\tan\theta},\ \frac{\tan\theta-\surd 3}{1+\surd 3\tan\theta}:$$

for $\tan 60^\circ = \surd 3$ and $\tan 120^\circ = -\surd 3$.

$$(3)\quad \tan\theta + \cot\theta' = \frac{\cos(\theta-\theta')}{\cos\theta\sin\theta'}.$$

$$(4)\quad \tan\theta - \cot\theta' = \frac{\cos(\theta+\theta')}{\cos\theta\sin\theta}.$$

$$(5)\quad \frac{\tan\theta+\tan\theta'}{\tan\theta-\tan\theta'} = \frac{\sin(\theta+\theta')}{\sin(\theta-\theta')}.$$

$$(6)\quad \frac{\sin\theta+\sin\theta'}{\cos\theta+\cos\theta'} = \tan\left(\frac{\theta+\theta'}{2}\right).$$

$$(7)\quad \frac{\sin\theta-\sin\theta'}{\cos\theta+\cos\theta'} = \tan\left(\frac{\theta-\theta'}{2}\right).$$

$$(8)\quad \frac{\sin\theta+\sin\theta'}{\sin\theta-\sin\theta'} = \frac{\tan\left(\frac{\theta+\theta'}{2}\right)}{\tan\left(\frac{\theta-\theta'}{2}\right)}.$$

Definition of the secant and cosecant.

793. The reciprocal of the cosine of an angle is called its *secant*, and the reciprocal of the sine of an angle is called its *cosecant*.

If θ be an angle, then the secant of θ or $\sec\theta = \frac{1}{\cos\theta}$: and the cosecant of θ or $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$.

If BAC (Fig. in Art. 786) be an angle, and if from any point P in one of the lines containing it we draw a perpendicular PM upon the other, then the ratio $\frac{AP}{AM}$ is the secant of BAC or θ, and the ratio $\frac{AP}{PM}$ is its cosecant*.

The terms *secant* and *cosecant* are not very commonly used, it being equally convenient to use the actual reciprocals of the sine and cosine in most of the expressions in which they would otherwise occur.

Their changes of sign.

794. The changes of sign of the secant and cosecant are those of the cosine and sine respectively. Likewise the secant of θ in-

* The secant and cosecant were formerly defined as lines, and not as ratios: thus, if BP (Figure in Note, Art. 785), be the arc of a circle subtending the angle θ at the centre, and if BT and Ct be respectively drawn from B and from the extremity C, of the quadrant BPC, tangents to the circle, then AT was called the *secant*, and At the *cosecant* of the angle BAP or θ.

creases from 1 when $\theta = 0$, to infinity when $\theta = \frac{\pi}{2}$: it then changes its sign and decreases from *infinity* to -1 when $\theta = \pi$: from $\theta = \pi$ to $\theta = \frac{3\pi}{2}$ it increases from -1 to *infinity*: and from $\theta = \frac{3\pi}{2}$ to $\theta = 2\pi$, it decreases from *infinity* to 1: the cosecant of θ is *infinity* when $\theta = 0$, and 1 when $\theta = \frac{\pi}{2}$: it becomes infinity again when $\theta = \pi$, where its sign changes: when $\theta = \frac{3\pi}{2}$, it is -1, and when $\theta = 2\pi$, it is infinite and negative.

795. The relations of the tangent and cotangent with the secant and cosecant are expressed by the equations (Art. 787, No. 10), The relations of the tangent and cotangent with the secant and cosecant.

$$\sqrt{(1 + \tan^2 \theta)} = \sec \theta,$$
$$\sqrt{(1 + \cot^2 \theta)} = \operatorname{cosec} \theta.$$

796. The *versed sine* of an angle is the abbreviated expression for $1 - \cos \theta$: it is written $vers\, \theta$. The versed sine of an angle.

If BAP be an angle, and if PM be drawn perpendicular to AB (Fig. in Note, Art. 785), then the ratio $\frac{BM}{AB}$ is the *versed sine* of the angle BAP *.

It never changes its sign, and the limits of its values are 0 when $\theta = 0$, and 2 when $\theta = \pi$: it increases whilst θ increases from 0 to π, and it diminishes whilst θ increases from π to 2π.

The use of this term is now almost entirely abandoned.

* The versed sine was formerly defined as the line BM, intercepted between the beginning B of the arc and the perpendicular PM drawn through its extremity: it was sometimes called the *sagitta* of an arc, for if the arc PB was doubled, and PM produced to make a complete chord, then BM would be the position of the arrow upon the stretched bow: it is chiefly with a view to make the writings of the older mathematicians intelligible, that it is expedient to refer to terms and definitions which have now fallen into disuse.

CHAPTER XXIX.

ON THE CONSTRUCTION OF A CANON OF SINES AND COSINES, TANGENTS AND COTANGENTS, SECANTS AND COSECANTS.

What is meant by a canon of sines, cosines, &c.

797. We have hitherto considered sines and cosines, tangents and cotangents as possessing determinate values for determinate angles, without attempting to assign them, except in the cases of angles of 45^0, 30^0 and 18^0, and their successive binary multiples and submultiples (Arts. 766, 709, 777): in the present Chapter we shall proceed to shew in what manner their numerical values may be determined generally for every minute and degree of the quadrant, with a view to the construction of a *Table* or *Canon*, in which those successive values may be registered: for in the ordinary applications of Trigonometry, the sine or cosine, corresponding to a given angle, and conversely, the angle corresponding to a given sine or cosine, are not found, by the actual calculation of their values, but always by reference to such a Table.

To find the sine and cosine of 1′.

798. As the basis of our enquiries, we shall begin with the calculation of the numerical values of the sine and cosine of 1′.

If, in the formula (Art. 776),

$$\sin\frac{\theta}{2} = \tfrac{1}{2}\sqrt{(1+\sin\theta)} - \tfrac{1}{2}\sqrt{(1-\sin\theta)},$$

we replace θ successively by 30^0 {whose sine is $\frac{1}{2}$, (Art. 766)}, $\frac{30^0}{2}$, $\frac{30^0}{2^2}$, $\frac{30^0}{2^3}$, and so on as far as $\frac{30^0}{2^{11}}$, we shall find,

$$\sin\frac{30^0}{2} = \tfrac{1}{2}\sqrt{(1+\tfrac{1}{2})} - \tfrac{1}{2}\sqrt{(1-\tfrac{1}{2})} = .258819 = s_1,$$

$$\sin\frac{30^0}{2^2} = \tfrac{1}{2}\sqrt{(1+s_1)} - \tfrac{1}{2}\sqrt{(1-s_1)} = .1305262 = s_2,$$

$$\sin\frac{30^0}{2^3} = \tfrac{1}{2}\sqrt{(1+s_2)} - \tfrac{1}{2}\sqrt{(1-s_2)} = .0654031 = s_3,$$

. . . .

$$\sin\frac{30^0}{2^{11}} = \tfrac{1}{2}\sqrt{(1+s_{10})} - \tfrac{1}{2}\sqrt{(1-s_{10})} = s_{11} = .000255625.$$

But $\frac{30^0}{2^{11}} = \frac{30 \times 60 \times 1'}{2^{11}} = \frac{15 \times 15 \times 1'}{2^8} = \frac{225}{256} \times 1'$, and is therefore the first of the successive binary submultiples of 30^0 which is less than $1'$: and inasmuch as the sines of very small angles increase and diminish very nearly in the same proportion with the angles themselves*, it follows that

$$\sin \frac{225}{256} \times 1' = \frac{225}{256} \sin 1' = .000255625,$$

and therefore

$$\sin 1' = \frac{256}{225} \times .000255625 = .0002908882 \text{ nearly.}$$

The corresponding value of the cosine of $1'$ derived from the equation

$$\cos 1' = \sqrt{(1 - \sin^2 1')}$$

gives us

$$\cos 1' = .9999999577.$$

799. The knowledge of the sine and cosine of $1'$ of a degree, will form the basis of our calculation of the sines and cosines of all angles differing from each other by $1'$, between an angle of $1'$ and 90^0.

Given the sine and cosine of $1'$, to construct a canon of sines and cosines.

For this purpose we make use of the formulæ, (Art. 774),

$$\sin (n + 1)\,\theta = 2 \cos \theta \sin n\theta - \sin (n - 1)\,\theta,$$

$$\cos (n + 1)\,\theta = 2 \cos \theta \cos n\theta - \cos (n - 1)\,\theta.$$

* The truth of this proposition may be inferred from the equation

$$\sin 2\theta = 2 \cos \theta \sin \theta \quad \text{or} \quad \frac{\sin 2\theta}{\sin \theta} = 2 \cos \theta,$$

where, when θ is very small, $\cos \theta$ is very nearly equal to 1, and therefore the sine is almost exactly doubled when the angle is doubled, and conversely: thus, $\cos \frac{30^0}{2^{11}} = .9999999674$, which differs from 1 by .0000000326 only: again, if we recur to the calculation in the text, we shall find

$$\sin \frac{30^0}{2^9} = .0010224959,$$

$$\sin \frac{30^0}{2^{10}} = .0005112482,$$

$$\sin \frac{30^0}{2^{11}} = .0002556254,$$

from whence it will be seen, for very small angles, how nearly the sines are bisected, when the angles are so.

If we replace θ by $1'$, and substitute successively the natural numbers 1, 2, 3, 4, &c. for n, we shall get

$$\sin 2' = 2 \cos 1' \sin 1' \qquad = .0005817764,$$
$$\sin 3' = 2 \cos 1' \sin 2' - \sin 1' = .0008726645,$$
$$\sin 4' = 2 \cos 1' \sin 3' - \sin 2' = .0011635526,$$
$$\sin 5' = 2 \cos 1' \sin 4' - \sin 3' = .0014544406,$$

....

$$\cos 2' = 2 \cos 1' \cos 1' - \cos 0 = .9999998308,$$
$$\cos 3' = 2 \cos 1' \cos 2' - \cos 1' = .9999996192,$$
$$\cos 4' = 2 \cos 1' \cos 3' - \cos 2' = .9999993231,$$
$$\cos 5' = 2 \cos 1' \cos 4' - \cos 3' = .9999989423,$$

....

Not necessary to proceed beyond 45°.

In this manner we might proceed as far as the sine and cosine of 45°, in proceeding beyond which we shall find the same values of sines and cosines which have been previously determined: for inasmuch as $\sin(45^\circ + \theta) = \cos(45^\circ - \theta)$ and $\cos(45^\circ + \theta) = \sin(45^\circ - \theta)$, it will follow that the sines of the angles in the series ascending from 45° will be severally equal to the cosines of the *complements* of angles in the descending series which have been already determined, and conversely: it thus becomes necessary to construct the *canon* of sines and cosines as far as 45°, and no further.

Beyond 30° the process of calculation may be greatly simplified.

800. But it will be observed that the processes of calculation, which are founded upon the formulæ in the last Article, involve the formation of a very laborious product, which it is expedient to supersede, when practicable, by others which require the more simple and expeditious operations of addition and subtraction only: one of the most convenient of these is the formula

$$\sin(30^\circ + \theta) = \cos\theta - \sin(30^\circ - \theta)^*,$$

by which the sines of the angles from 30° to 60° may be calculated by the mere subtraction of the sines of $(30^\circ - \theta)$ (where θ may have every value from $1'$ to 30°), from the cosine of θ: and since

$$\sin(30^\circ + \theta) = \cos(60^\circ - \theta),$$

the determination of the series of sines from 30° to 60° will give us the corresponding series of cosines in an inverse order.

* For $\sin(30^\circ + \theta) + \sin(30^\circ - \theta) = 2 \sin 30^\circ \cos\theta = \cos\theta$, since $2 \sin 30^\circ = 1$.

801. In the formulæ which we have given, we have calculated the values of the sines and cosines, at least as far as 30^0, independently of each other, from the ascertained values of the sine and cosine of $1'$, and consequently we may use the values of the sine and cosine of any assigned angle θ, to *test* the accuracy of the calculation, by substituting them in the equation $\cos^2\theta + \sin^2\theta = 1$: for if this equation be not satisfied, the values of one or both of them are necessarily incorrect: thus if we take the values of the sine and cosine of $5'$, which are found in Art. 799, we shall find

Process of verifying the correctness of the results obtained.

$$\cos^2 5' = .99999788460111872929,$$

$$\sin^2 5' = .00000211539745892836,$$

and

$$\cos^2 5' + \sin^2 5' = .99999999999857765765,$$

a number which differs from 1 by a quantity less than

$$.0000000000015,$$

a discrepancy which is referrible to the influence of terms in the calculated values of $\cos 5'$ and $\sin 5'$, which are necessarily omitted, as being beyond the 10^{th} place of decimals, to which the registered values are limited.

802. The methods of calculating the successive terms of a canon of sines and cosines, are *methods of continuation*, where an error committed in the determination of any one of them is transmitted to all those which succeed it: and it is in order to arrest the continued propagation of errors, as well as to verify the correctness of the calculations, at different points of their progress, when no such errors exist, that it is usual to interpose, as *stops*, the values of any such terms in the series as can be determined by independent methods. Such are the sines and cosines of 45^0, 30^0 and 18^0, and their binary submultiples, or of the sum and difference of any angles in the series thus formed*.

Process of verification by the interposition of stops or of values calculated by other methods.

803. A canon of tangents may be formed from a canon of sines and cosines, by dividing the sines by the cosines: and a canon of cotangents may be similarly formed by dividing the cosines by the sines. If however we have found the tangents

Formation of a canon of tangents and cotangents.

* We thus get

$$\sin(18^0 - 15^0) = \sin 3^0 = \sin 18^0 \cos 15^0 - \cos 18^0 \sin 15^0 = .0523360.$$

between 1′ and 45°, we may form the tangents between 45° and 90°, or the cotangents between 1′ and 45°, by means of the formula

$$\tan(45^\circ + \theta) = 2\tan 2\theta + \tan(45^\circ - \theta)\,*,$$

which involves the most simple operations only.

Thus if we replace θ successively by 1′, 2′, 3′..., we get

$$\tan 45^\circ.\,1' = 2\tan 2' + \tan 44^\circ.\,59',$$

$$\tan 45^\circ.\,2' = 2\tan 4' + \tan 44^\circ.\,58',$$

$$\tan 45^\circ.\,3' = 2\tan 6' + \tan 44^\circ.\,57',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

A canon of secants and cosecants.

804. A canon of secants and cosecants may be formed immediately from a canon of cosines and sines, the secant being the reciprocal of the cosine, and the cosecant the reciprocal of the sine: or much more rapidly, from a canon of tangents and cotangents by means of the formulæ

$$\operatorname{cosec}\theta = \tfrac{1}{2}\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\dagger$$

$$\sec\theta = \tfrac{1}{2}\left\{\tan\left(45^\circ + \frac{\theta}{2}\right) + \cot\left(45^\circ + \frac{\theta}{2}\right)\right\}.$$

Formulæ of verification.

805. Formulæ of *verification* are equations between the sines and cosines, tangents and cotangents of different angles, which if satisfied by their values, as given in the canon, or not, would verify their correctness, or the contrary.

Such are the equations:

(1) $\operatorname{Sin}\theta = \sin(60^\circ + \theta) - \sin(60^\circ - \theta)$.

(2) $\operatorname{Sin}\theta = \cos(30^\circ - \theta) - \cos(30^\circ + \theta)$.

(3) $\operatorname{Cos}\theta = \cos(60^\circ + \theta) + \cos(60^\circ - \theta)$.

(4) $\operatorname{Cos}\theta = \sin(30^\circ + \theta) + \sin(30^\circ - \theta)$.

(5) $\operatorname{Tan}\theta = \cot\theta - 2\cot 2\theta$.

* For $\tan(45^\circ + \theta) - \tan(45^\circ - \theta) = \dfrac{\tan 45^\circ + \tan\theta}{1 - \tan 45^\circ\tan\theta} - \dfrac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta}$

$= \dfrac{1 + \tan\theta}{1 - \tan\theta} - \dfrac{1 - \tan\theta}{1 + \tan\theta}$ (for $\tan 45^\circ = 1$)

$= \dfrac{4\tan\theta}{1 - \tan^2\theta} = 2\tan 2\theta$ {for $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ (Art. 791)}.

† For, by Note, Art. 791, we find

$$\tan\theta = \operatorname{cosec} 2\theta - \cot 2\theta,$$
$$\cot\theta = \operatorname{cosec} 2\theta + \cot 2\theta,$$

and therefore, by adding, we get

$$\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta.$$

(6) $\operatorname{Sin}\ \theta + \sin(72^\circ + \theta) - \sin(72^\circ - \theta)$
$$= \sin(36^\circ + \theta) - \sin(36^\circ - \theta).$$

(7) $\operatorname{Sin}(90^\circ - \theta) + \sin(18^\circ + \theta) + \sin(18^\circ - \theta)$
$$= \sin(54^\circ + \theta) + \sin(54^\circ - \theta).$$

(8) $\operatorname{Cos}\ \theta + \cos(72^\circ + \theta) + \cos(72^\circ - \theta)$
$$= \cos(36^\circ + \theta) + \cos(36^\circ - \theta).$$

(9) $\operatorname{Cos}(90^\circ - \theta) + \cos(18^\circ - \theta) - \cos(18^\circ - \theta)$
$$= \cos(54^\circ - \theta) - \cos(54^\circ + \theta)^*.$$

Such formulæ, however, are better suited to verify the correctness of canons of sines and cosines, tangents and cotangents, already formed and calculated, than to aid us in providing against the intrusion and transmission of errors in the progress of their formation.

* These equations are easily verified by developing the sines and cosines of the sums or differences of the angles which they involve, and substituting the numerical values of the sines and cosines of 30^0 and 18^0, or their multiples.

CHAPTER XXX.

ON DEMOIVRE'S FORMULA, AND THE EXPRESSION, BY MEANS OF IT, OF THE ROOTS OF 1 AND OF $a \pm b\sqrt{-1}$.

Demoivre's formula.

806. If we multiply together two such expressions as

$$\cos\phi + a\sin\phi \text{ and } \cos\theta + a\sin\theta,$$

we shall find

$$(\cos\phi + a\sin\phi)(\cos\theta + a\sin\theta)$$
$$= \cos\phi\cos\theta + a^2\cos\phi\cos\theta + a(\cos\phi\sin\theta + \sin\phi\cos\theta).$$

If we replace a by $\sqrt{-1}$, and therefore a^2 by -1, this equation becomes

$$(\cos\phi + \sqrt{-1}\sin\phi)(\cos\theta + \sqrt{-1}\sin\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$+ \sqrt{-1}(\cos\phi\sin\theta + \sin\phi\cos\theta) = \cos(\phi+\theta) + \sqrt{-1}\sin(\phi+\theta).$$

(Art. 770.)

If we make $\phi = \theta$, we get

$$(\cos\theta + \sqrt{-1}\sin\theta)^2 = \cos 2\theta + \sqrt{-1}\sin 2\theta.$$

If we make $\phi = 2\theta$, we get

$$(\cos 2\theta + \sqrt{-1}\sin 2\theta)(\sin\theta + \sqrt{-1}\sin\theta) = (\cos\theta + \sqrt{-1}\sin\theta)^3$$
$$= \cos 3\theta + \sqrt{-1}\sin 3\theta.$$

If we make $\phi = 3\theta$, we get

$$(\cos 3\theta + \sqrt{-1}\sin 3\theta)(\cos\theta + \sqrt{-1}\sin\theta) = (\cos\theta + \sqrt{-1}\sin\theta)^4$$
$$= \cos 4\theta + \sqrt{-1}\sin 4\theta.$$

The law of formation of this formula being thus indicated, we may assume

$$(\cos\theta + \sqrt{-1}\sin\theta)^{n-1} = \cos(n-1)\theta + \sqrt{-1}\sin(n-1)\theta;$$

and making $\phi = (n-1)\theta$, and therefore $\phi + \theta = n\theta$, we get

$$\{\cos(n-1)\theta + \sqrt{-1}\sin(n-1)\theta\}(\cos\theta + \sqrt{-1}\sin\theta)$$
$$= \cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^{n-1}(\cos\theta + \sqrt{-1}\sin\theta)$$
$$= (\cos\theta + \sqrt{-1}\sin\theta)^n.$$

It thus appears, that if the formula

$$\cos(n-1)\theta + \sqrt{-1}\sin(n-1)\theta = (\cos\theta + \sqrt{-1}\sin\theta)^{n-1}$$

be true, then the formula

$$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^{n}$$

is necessarily true (Art. 447): and inasmuch as it has been shewn to be true when n is 2, 3, 4, it is necessarily true when n is 5, 6, ... and so on, for any whole number whatsoever.

In a similar manner it may be shewn that

$$\cos n\theta - \sqrt{-1}\sin n\theta = (\cos\theta - \sqrt{-1}\sin\theta)^{n}.$$

Again, if, in the formulæ just established, we replace θ by $\frac{\theta}{m}$ (for θ may be any angle, great or small, positive or negative*), we get

$$\cos\frac{n\theta}{m} + \sqrt{-1}\sin\frac{n\theta}{m} = \left(\cos\frac{\theta}{m} + \sqrt{-1}\sin\frac{\theta}{m}\right)^{n}:$$

and since

$$\cos\theta + \sqrt{-1}\sin\theta = \left(\cos\frac{\theta}{m} + \sqrt{-1}\sin\frac{\theta}{m}\right)^{m};$$

and therefore

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{m}} = \cos\frac{\theta}{m} + \sqrt{-1}\sin\frac{\theta}{m},$$

it follows that

$$\cos\frac{n\theta}{m} + \sqrt{-1}\sin\frac{n\theta}{m} = \{(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{m}}\}^{n}$$

$$= (\cos\theta + \sqrt{-1}\sin\theta)^{\frac{n}{m}}.$$

Also, since

$$(\cos\theta + \sqrt{-1}\sin\theta)(\cos\theta - \sqrt{-1}\sin\theta) = \cos^2\theta + \sin^2\theta = 1,$$

it follows that

$$\cos\theta + \sqrt{-1}\sin\theta = (\cos\theta - \sqrt{-1}\sin\theta)^{-1},$$

and therefore

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{n}{m}} = (\cos\theta - \sqrt{-1}\sin\theta)^{-\frac{n}{m}}.$$

* For if in the formula

$$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^{n}$$

we replace θ by $-\theta$, we get

$$\cos n\theta - \sqrt{-1}\sin n\theta = (\cos\theta - \sqrt{-1}\sin\theta)^{n},$$

a conclusion which may be otherwise shewn to be true.

$$\text{But } \cos\frac{n\theta}{m} + \sqrt{-1}\sin\frac{n\theta}{m} = \cos(\theta + \sqrt{-1}\sin\theta)^{\frac{n}{m}}$$

$$= \cos(\theta - \sqrt{-1}\sin\theta)^{-\frac{n}{m}}:$$

and, inasmuch as $\cos\frac{n\theta}{m} + \sqrt{-1}\sin\frac{n\theta}{m}$ becomes

$$\cos\frac{n\theta}{m} - \sqrt{-1}\sin\frac{n\theta}{m},$$

when $\frac{n}{m}$ is replaced by $-\frac{n}{m}$, it follows that

$$\cos\frac{n\theta}{m} - \sqrt{-1}\sin\frac{n\theta}{m} = (\cos\theta + \sqrt{-1}\sin\theta)^{-\frac{n}{m}},$$

consequently

$$\cos\frac{n\theta}{m} \pm \sqrt{-1}\sin\frac{n\theta}{m} = (\cos\theta \mp \sqrt{-1}\sin\theta)^{-\frac{n}{m}}.$$

It thus appears that the formula

$$\cos n\theta \pm \sqrt{-1}\sin\theta = (\cos\theta \pm \sqrt{-1}\sin\theta)^{n}\,*$$

is true for all values of the index. It is known, from the name of its discoverer, as Demoivre's formula, and constitutes one of the most important propositions in the whole range of analysis.

Exponential expression for the sine and cosine of θ.

807. If, in virtue of the preceding proposition, we should suppose a^{θ} to be equal to

$$\cos\theta + \sqrt{-1}\sin\theta,$$

and therefore $a^{n\theta} = \cos n\theta + \sqrt{-1}\sin n\theta$, we might treat

$$a^{n\theta} \text{ and } \cos n\theta + \sqrt{-1}\sin n\theta$$

as possessing common properties, and as immediately convertible with each other: we should thus find

$$\cos\theta + \sqrt{-1}\sin\theta = a^{\theta}$$

$$\cos\theta - \sqrt{-1}\sin\theta = a^{-\theta},$$

* The demonstration in the text is dependent upon the properties of indices, by the aid of which we are enabled to shew that

$$\cos n\theta \pm \sqrt{-1}\sin n\theta \text{ and } (\cos\theta \pm \sqrt{-1}\sin\theta)^{n}$$

are in every respect convertible into each other: and as the general properties of indices are referrible for their authority to the "principle of the permanence of equivalent forms," (Art. 631), so likewise must the formula under consideration be ultimately referrible to the same principle for its establishment: but the properties of indices being once admitted as algebraical truths, we refer to them as furnishing the immediate authority for other truths deducible by means of them, and not to the fundamental principles upon which they rest in common.

and therefore, adding and subtracting, and also dividing their sum by 2 and their difference by $2\sqrt{-1}$, we should find

$$\cos\theta = \frac{a^\theta + a^{-\theta}}{2},$$

$$\sin\theta = \frac{a^\theta - a^{-\theta}}{2\sqrt{-1}}:$$

and again, since

$$\cos n\theta + \sqrt{-1}\sin n\theta = a^{n\theta},$$

$$\cos n\theta - \sqrt{-1}\sin n\theta = a^{-n\theta},$$

we should find, in like manner,

$$\cos n\theta = \frac{a^{n\theta} + a^{-n\theta}}{2},$$

and

$$\sin n\theta = \frac{a^{n\theta} - a^{-n\theta}}{2\sqrt{-1}}.$$

808. The exponential expressions for the cosine and sine of an angle θ, which are given in the last Article, will be found to satisfy the fundamental equations for those quantities, *whatever be the value of a*: for

They satisfy the fundamental equations for the sine and cosine for all values of a.

$$\cos^2\theta + \sin^2\theta = \left(\frac{a^\theta + a^{-\theta}}{2}\right)^2 + \left(\frac{a^\theta - a^{-\theta}}{2\sqrt{-1}}\right)^2$$

$$= \frac{a^{2\theta} + 2 + a^{-2\theta}}{4} - \frac{a^{2\theta} - 2 + a^{-2\theta}}{4} = 1, \quad \text{Art. 759,}$$

and also

$$\cos\theta = \frac{a^{-\theta} + a^\theta}{2} = \cos\theta,$$

$$\sin -\theta = \frac{a^{-\theta} - a^\theta}{2\sqrt{-1}} = -\sin\theta, \quad \text{Art. 762,}$$

809. It should be observed that the definition of the sine and cosine of an angle is dependent upon the goniometrical angle and its measure no further than it serves to determine the geometrical angle, and therefore will remain the same whether the measure of the goniometrical angle be the ratio of the arc to the radius, or of the arc to the diameter, or any other ratio whatever which increases or diminishes in the same proportion with it: and it is for this reason, that if the value of θ, for a given geometrical angle, be indeterminate, the value of a will be indeter-

The value of a dependent upon the assumed measure of an angle: if the ordinary measure be taken, it may be replaced by $e^{\sqrt{-1}}$.

minate likewise: but if we agree to assume, as the measure of angles, the ratio of the arc to the radius, as adopted in Art. 746, the analytical value of a (for in no case does it possess an arithmetical value) will be shewn in a subsequent Chapter to be determinate and equal to $e^{\sqrt{-1}}$, where $e = 2.7182818$ is the base of Napierian logarithms: replacing a by this value, we shall find

$$\cos\theta = \frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2}, \quad \sin\theta = \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{2\sqrt{-1}},$$

expressions, which comprehend all the conditions which we have imposed upon the sine and cosine of an angle and its measure, and from which we may very readily deduce all the formulæ which are given in Chapter XXVII, independently of their geometrical definition.

There are n different values of θ which give the same value of $\cos n\theta \pm \sqrt{-1}\sin n\theta$, but different values of $\cos\theta \pm \sqrt{-1}\sin\theta$.

810. If in the equation

$$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n$$

the values of $n\theta$ are to be determined from the values of its cosine and sine, then there are n different values of θ which give different values of

$$\cos\theta + \sqrt{-1}\sin\theta,$$

and which equally satisfy the equation.

For, inasmuch as the terms of the two series

$$n\theta,\ 2\pi + n\theta,\ 4\pi + n\theta, \ldots\ 2(n-1)\pi + n\theta,\ \ 2n\pi + n\theta,$$

$$\pi - n\theta,\ 3\pi - n\theta,\ 5\pi - n\theta, \ldots\ (2n-1)\pi - n\theta,\ (2n+1)\pi - n\theta,$$

and no others, are *equisinal*, (Art. 780), and those of the two series

$$n\theta,\ \ 2\pi + n\theta,\ \ 4\pi + n\theta, \ldots\ \ 2(n-1)\pi + n\theta,\ \ 2n\pi + n\theta,$$

$$-n\theta,\ \ 2\pi - n\theta,\ \ 4\pi - n\theta, \ldots\ \ 2(n-1)\pi - n\theta,\ \ 2n\pi - n\theta,$$

and no others, are *equicosinal*, (Art. 781), it follows that the terms of the series

$$n\theta,\ \ 2\pi + n\theta,\ \ 4\pi + n\theta, \ldots\ \ 2(n-1)\pi + n\theta,\ \ 2n\pi + n\theta,$$

and no others, are simultaneously *equisinal* and *equicosinal*, and therefore severally give the same value of the expression

$$\cos n\theta + \sqrt{-1}\,.\sin n\theta,$$

when both $\cos n\theta$ and $\sin n\theta$ are given.*

* If $\cos n\theta = a$ and $\sin n\theta = b$, then $a^2 + b^2 = 1$: if one be given therefore, the other is determined in magnitude, though not in sign.

It is the period of n terms

$$\theta,\ \frac{2\pi}{n}+\theta,\ \frac{4\pi}{n}+\theta,\ldots\ \frac{2(n-1)}{n}+\theta,$$

and no others, which give *different* values of $\cos\theta$ and $\sin\theta$, one or both of them, and therefore different values of

$$\cos\theta+\sqrt{-1}\,\sin\theta,$$

but which give the *same* values to both the terms of the expression

$$\cos n\theta+\sqrt{-1}\,\sin n\theta:$$

they form therefore the values of θ, and the only values of θ, which satisfy the equation

$$(\cos\theta+\sqrt{-1}\,\sin\theta)^n=\cos n\theta+\sqrt{-1}\,\sin n\theta.$$

Any one therefore of the n following expressions

$$(1)\qquad \cos\theta+\sqrt{-1}\,\sin\theta,$$

$$(2)\qquad \cos\left(\frac{2\pi}{n}+\theta\right)+\sqrt{-1}\,\sin\left(\frac{2\pi}{n}+\theta\right),$$

$$(3)\qquad \cos\left(\frac{4\pi}{n}+\theta\right)+\sqrt{-1}\,\sin\left(\frac{4\pi}{n}+\theta\right),$$

....

$$(n-1)\quad \cos\left\{\frac{2(n-2)\pi}{n}+\theta\right\}+\sqrt{-1}\,\sin\left\{\frac{2(n-2)\pi}{n}+\theta\right\},$$

$$(n)\qquad \cos\left\{\frac{2(n-1)\pi}{n}+\theta\right\}+\sqrt{-1}\,\sin\left\{\frac{2(n-1)\pi}{n}+\theta\right\},$$

when raised to the n^{th} power, will give the same value of the expression

$$\cos n\theta+\sqrt{-1}\,\sin n\theta$$

when $n\theta$ is to be determined from the value of its sine and cosine.

Inasmuch as $\cos(2\pi-\phi)=\cos\phi$ and $\sin(2\pi-\phi)=-\sin\phi$, it follows that

$$\cos\left\{\frac{2(n-r)\pi}{n}+\theta\right\}+\sqrt{-1}\,\sin\left\{\frac{2(n-r)\pi}{n}+\theta\right\}$$

$$=\cos\left(\frac{2r\pi}{n}-\theta\right)-\sqrt{-1}\,\sin\left(\frac{2r\pi}{n}-\theta\right);$$

and the preceding series of n different values may be reduced to equivalent forms, which involve no angle exceeding 180^0.

Determination of the n roots of 1. 811. Let it be required to express the n roots of 1: or in other words, to solve the equation

$$x^n - 1 = 0.$$

If, in the equation

$$\left\{\cos\left(\frac{2r\pi}{n}+\theta\right)+\sqrt{-1}\,\sin\left(\frac{2r\pi}{n}+\theta\right)\right\}^n = \cos n\theta + \sqrt{-1}\,\sin n\theta,$$

we make $\theta = 0$, and therefore $\cos n\theta = 1$ and $\sin n\theta = 0$, we get

$$\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\,\sin\frac{2r\pi}{n}\right)^n = 1,$$

and therefore

$$\cos\frac{2r\pi}{n}+\sqrt{-1}\,\sin\frac{2r\pi}{n} = (1)^{\frac{1}{n}},$$

in which r may be made 0, 1, 2,...$(n-1)$ successively, giving n results and *no more*, which are *different* from each other: these are the n roots of 1, or the n roots of the equation

$$x^n - 1 = 0.$$

The cube roots of 1. 812. Thus, let it be required to express the cube roots of 1. In this case, $n = 3$ and the roots are expressed by

(1) $\cos 0 + \sqrt{-1}\,\sin 0 \quad = 1.$

(2) $\cos\frac{2\pi}{3}+\sqrt{-1}\,\sin\frac{2\pi}{3} = \cos 120^0 + \sqrt{-1}\,\sin 120^0 = \frac{-1+\sqrt{3}\sqrt{-1}}{2}.$

(3) $\cos\frac{4\pi}{3}+\sqrt{-1}\,\sin\frac{4\pi}{3} = \cos\frac{2\pi}{3} - \sqrt{-1}\,\sin\frac{2\pi}{3}$

$$= \cos 120^0 - \sqrt{-1}\,\sin 120^0 = \frac{-1-\sqrt{3}\sqrt{-1}}{2}.$$

These results agree with those given in Art. 669.

The biquadratic roots of 1. 813. Let it be required to express the biquadratic roots of 1. In this case $n = 4$, and the roots are expressed by

(1) $\text{Cos}\, 0 + \sqrt{-1}\,\sin 0 = 1.$

(2) $\text{Cos}\,\frac{2\pi}{4}+\sqrt{-1}\,\sin\frac{2\pi}{4} = 0 + \sqrt{-1} = \sqrt{-1}.$

(3) $\text{Cos}\,\frac{4\pi}{4} + \sqrt{-1}\,\sin\frac{4\pi}{4} = -1.$

(4) $\text{Cos}\,\frac{6\pi}{4} + \sqrt{-1}\,\sin\frac{6\pi}{4} = \cos\frac{2\pi}{4} - \sqrt{-1}\,\sin\frac{2\pi}{4} = -\sqrt{-1}.$

These results agree with those given in Art. 695.

814. Let it be required to express the quinary roots of 1. In this case $n = 5$, and the several roots are expressed by The quinary roots of 1.

(1) $\text{Cos}\,0 + \sqrt{-1}\,\sin 0 = 1.$

(2) $\text{Cos}\,\frac{2\pi}{5} + \sqrt{-1}\,\sin\frac{2\pi}{5} = \cos 72^0 + \sqrt{-1}\,\sin 72^0 = .309 + \sqrt{-1} \times .951.$

(3) $\text{Cos}\frac{4\pi}{5} + \sqrt{-1}\,\sin\frac{4\pi}{5} = \cos 144^0 + \sqrt{-1}\,\sin 144^0$

$= -\cos 36^0 + \sqrt{-1}\,\sin 36^0 = -.809 + \sqrt{-1} \times .587.$

(4) $\text{Cos}\,\frac{2\pi}{5} - \sqrt{-1}\,\sin\frac{2\pi}{5} = \cos 72^0 - \sqrt{-1}\,\sin 72^0 = .309 - \sqrt{-1} \times .951.$

(5) $\text{Cos}\,\frac{4\pi}{5} - \sqrt{-1}\,\sin\frac{4\pi}{5} = -\cos 36^0 - \sqrt{-1}\,\sin 36^0 = -.809 - \sqrt{-1} \times .587.$

These results agree with those given in Art. 707.

815. Let it be required to express the septenary roots of 1. In this case $n = 7$, and the roots are The septenary roots of 1.

(1) $\text{Cos}\,0 + \sqrt{-1}\,\sin 0 = 1.$

(2) $\text{Cos}\frac{2\pi}{7} + \sqrt{-1}\,\sin\frac{2\pi}{7} = \cos 51^{\circ}.26' + \sqrt{-1}\,\sin 51^{\circ}.26'$

$= .623 + \sqrt{-1} \times .782.$

(3) $\text{Cos}\frac{4\pi}{7} + \sqrt{-1}\,\sin\frac{4\pi}{7} = -\cos\frac{3\pi}{7} + \sqrt{-1}\,\sin\frac{3\pi}{7}$

$= -\cos 77^{\circ}.9' + \sqrt{-1}\,\sin 77^{\circ}.9' = -.222 + \sqrt{-1} \times .975.$

(4) $\text{Cos}\,\frac{6\pi}{7} + \sqrt{-1}\,\sin\frac{6\pi}{7} = -\cos\frac{\pi}{7} + \sqrt{-1}\,\sin\frac{\pi}{7}$

$= -\cos 25^{\circ}.43' + \sqrt{-1}\,\sin 25^{\circ}.43' = -.893 + \sqrt{-1} \times .450.$

(5) $\text{Cos}\,\frac{8\pi}{7} + \sqrt{-1}\,\sin\frac{8\pi}{7} = -\cos\frac{\pi}{7} - \sqrt{-1}\,\sin\frac{\pi}{7} \ldots$

$= -.893 - \sqrt{-1} \times .450.$

(6) $\text{Cos}\,\frac{4\pi}{7} - \sqrt{-1}\,\sin\frac{4\pi}{7} = -\cos\frac{3\pi}{7} - \sqrt{-1}\,\sin\frac{3\pi}{7}\,\ldots$

$$= -.222 - \sqrt{-1} \times .975.$$

(7) $\text{Cos}\,\frac{2\pi}{7} - \sqrt{-1}\,\sin\frac{2\pi}{7} = \ldots\ldots\ldots\ldots\ldots .623 - \sqrt{-1} \times .782.$

The preceding Examples will be sufficient to shew how rapidly the roots of 1, *of any order*, may be found by means of a Canon of sines and cosines, in which their successive values are registered.

To express the n roots of -1.

816. Let it be required to express the n roots of -1: or, in other words, to solve the equation

$$x^n + 1 = 0.$$

If, in the equation

$$\left\{\cos\left(\frac{2r\pi}{n} + \theta\right) + \sqrt{-1}\,\sin\left(\frac{2r\pi}{n} + \theta\right)\right\}^n = \cos n\theta + \sqrt{-1}\,\sin n\theta,$$

we make $n\theta = \pi$, and therefore $\cos n\theta = -1$, and $\sin n\theta = 0$, we get

$$\left\{\cos\frac{(2r+1)\pi}{n} + \sqrt{-1}\,\sin\frac{(2r+1)}{n}\pi\right\}^n = -1,$$

and therefore

$$\cos\frac{(2r+1)\pi}{n} + \sqrt{-1}\,\sin\frac{(2r+1)\pi}{n} = (-1)^{\frac{1}{n}},$$

in which r may be $0, 1, 2 \ldots (n-1)$ successively, giving n results and *no more:* these are the n roots of -1, or the n roots of the equation

$$x^n + 1 = 0.$$

The n roots of -1 are included amongst the $2n$ roots of 1.

817. The n roots of -1 are the 2nd, 4th, 6th and $2n^{\text{th}}$ terms of the period of $2n$ roots of 1, arranged in their order, or the alternate roots of the equation

$$x^{2n} - 1 = 0.$$

For the 2nd, 4th, 6th, ... $2n^{\text{th}}$ roots of $x^{2n} - 1$ are

$$\cos\frac{2\pi}{2n} + \sqrt{-1}\,\sin\frac{2\pi}{2n},\ \cos\frac{6\pi}{2n} + \sqrt{-1}\,\sin\frac{6\pi}{2n},\ \ldots$$

$$\ldots \cos\frac{2(n-1)\pi}{2n} + \sqrt{-1}\,\sin\frac{2(n-1)\pi}{2n},$$

or

$$\cos\frac{\pi}{n} + \sqrt{-1}\,\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n} + \sqrt{-1}\,\sin\frac{3\pi}{n}, \ldots$$

$$\ldots \cos\frac{(n-1)\pi}{n} + \sqrt{-1}\,\sin\frac{(n-1)\pi}{n},$$

which are also the n roots of -1.

818. Let it be required to express the complete values of a^n and $(-a)^n$, considered as derived respectively from a and $-a$, for all values of n. To express the complete values of a^n and $(-a)^n$.

Assume ρ to represent the *arithmetical* value of a^n or $(-a)^n$, which is independent of any sign of affection, and is the same therefore for both.

We then find

$$a^n = (1)^n\rho = (\cos 2nr\pi + \sqrt{-1}\,\sin 2nr\pi)\,\rho$$

$$(-a)^n = (-1)^n\rho = \{\cos(2r+1)\,n\pi + \sqrt{-1}\,\sin(2r+1)\,n\pi\}\,\rho,$$

where r is any number in the series $0, 1, 2, 3 \ldots\ldots$

If n be any whole number, $a^n = \rho$.

If n be an even whole number, $(-a)^n = \rho$: for in that case, $(2r+1)\,n\pi$ is an *even* multiple of π.

If n be an odd whole number, $(-a)^n = -\rho$: for in that case, $(2r+1)\,n\pi$ is an *odd* multiple of π.

If n be a rational fraction, having a denominator p: then the values of r may be limited to the terms of the series

$$0,\ 1,\ 2, \ldots p-1:$$

beyond these limits, whether the series be continued backwards or forwards, the same values of a^n or $(-a)^n$ recur.

If n be an irrational number (Art. 245), every *different* value of r, in the ascending series, will give a *different* value of $(a)^n$ or $(-a)^n$: and such values, therefore, never recur.

819. Let it be required to express the complete values of To express the complete values of $(a+b\sqrt{-1})^n$ and $(a-b\sqrt{-1})^n$

$$(a + b\sqrt{-1})^n \text{ and } (a - b\sqrt{-1})^n,$$

considered as derived respectively from

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1}.$$

If we make $\cos\theta = \dfrac{a}{\sqrt{(a^2+b^2)}}$, and

$$\sin\theta = \sqrt{(1-\cos^2\theta)} = \sqrt{\left\{1 - \frac{a^2}{a^2+b^2}\right\}} = \frac{b}{\sqrt{(a^2+b^2)}},$$

we get

$$\cos\theta + \sqrt{-1}\sin\theta = \frac{a}{\sqrt{(a^2+b^2)}} + \frac{b\sqrt{-1}}{\sqrt{(a^2+b^2)}},$$

and

$$\cos\theta - \sqrt{-1}\sin\theta = \frac{a}{\sqrt{(a^2+b^2)}} - \frac{b\sqrt{-1}}{\sqrt{(a^2+b^2)}}.$$

It follows, therefore, that

$$\sqrt{(a^2+b^2)}\{\cos\theta + \sqrt{-1}\sin\theta\} = a + b\sqrt{-1},$$

and

$$\sqrt{(a^2+b^2)}\{\cos\theta - \sqrt{-1}\sin\theta\} = a - b\sqrt{-1}.$$

But, inasmuch as the value of θ is derived from that of $\cos\theta$ and $\sin\theta$, it follows that

$$\frac{a}{\sqrt{(a^2+b^2)}} = \cos\theta = \cos(2r\pi + \theta),$$

$$\frac{b}{\sqrt{(a^2+b^2)}} = \sin\theta = \sin(2r\pi + \theta).$$

Consequently, if we make $(a^2+b^2)^{\frac{n}{2}} = \rho$, we get

$$(a + b\sqrt{-1})^n = \{\cos(2r\pi + \theta) + \sqrt{-1}\sin(2r\pi + \theta)\}^n \rho$$

$$= \{\cos n(2r\pi + \theta) + \sqrt{-1}\sin n(2r\pi + \theta)\}\rho,$$

$$(a - b\sqrt{-1})^n = \{\cos n(2r\pi + \theta) - \sqrt{-1}\sin n(2r\pi + \theta)\}\rho.$$

If we suppose $b = 0$ and therefore $\theta = 0$, these formulæ degenerate into those which are given in the last Article.

The expression $\cos\theta + \sqrt{-1}\sin\theta$ may in all cases be considered as a root of 1.

820. The conclusions, which are given in the preceding Articles, are of very great importance, and deserve the most careful attention of the student.

In the first place, in expressions such as

$$(\cos\theta \pm \sqrt{-1}\sin\theta)\rho,$$

we must consider $\cos\theta \pm \sqrt{-1}\sin\theta$ as a sign of affection of ρ, and as in no respect affecting its magnitude: for it may, in all cases, be shewn to coincide with one of the roots of 1, and to be therefore incapable of modifying the magnitude of the geometrical or other quantity into which it is multiplied (Art. 727): for, inasmuch as

$$\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n} = (1)^{\frac{1}{n}},$$

where r may be any whole number, and n any numerical mag-

nitude whatsoever, whether whole, fractional or irrational, it will follow that there is always some integral value of r, and, when n is not rational, some approximate fractional value of n, to be found, which will make $\frac{2r\pi}{n}$ equal to, or indefinitely nearly equal to, any assigned value of θ, and therefore to authorize us in considering $\cos\theta + \sqrt{-1}\sin\theta$ as in all cases coinciding with some one of the roots of 1, and therefore possessing their periodical properties.

In the second place it has been shewn that $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are equal to $\sqrt{(a^2+b^2)}(\cos\theta + \sqrt{-1}\sin\theta)$ and $\sqrt{(a^2+b^2)}(\cos\theta - \sqrt{-1}\sin\theta)$ respectively, and consequently that $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ respectively express the same arithmetical or geometrical magnitude $\sqrt{(a^2+b^2)}$, affected by the sign $\cos\theta+\sqrt{-1}\sin\theta$ $\left\{\text{where } \cos\theta = \frac{a}{\sqrt{(a^2+b^2)}} \text{ and } \sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}\right\}$ in one case, and by the sign $\cos\theta - \sqrt{-1}\sin\theta$ in the other: these conclusions will be found to be of fundamental importance in the interpretation of those signs, when applied to lines in geometry, and which will be given in the following Chapter.

$\sqrt{(a^2+b^2)}$ is the common arithmetical value of $a+b\sqrt{-1}$, and $a-b\sqrt{-1}$, which differ in affection only and not in magnitude.

821. Let it be required to express the complete values of

$$(a+b\sqrt{-1})^n + (a-b\sqrt{-1})^n,$$

and of

$$(a+b\sqrt{-1})^n - (a-b\sqrt{-1})^n.$$

To express the values of the sum or difference of $(a+b\sqrt{-1})^n$ and $(a-b\sqrt{-1})^n$

Adopting the expressions for $(a+b\sqrt{-1})^n$ and $(a-b\sqrt{-1})^n$, which are given in Art. 819, we find

$$\begin{aligned}&(a+b\sqrt{-1})^n + (a-b\sqrt{-1})^n\\ &= \rho\,\{\cos n\,(2r\pi+\theta) + \sqrt{-1}\sin n\,(2r\pi+\theta)\}\\ &+ \rho\,\{\cos n\,(2r\pi+\theta) - \sqrt{-1}\sin n\,(2r\pi+\theta)\}\\ &= 2\rho\cos n\,(2r\pi+\theta),\end{aligned}$$

and similarly

$$(a+b\sqrt{-1})^n - (a-b\sqrt{-1})^n = 2\rho\sqrt{-1}\sin(2r\pi+\theta),$$

where $\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}}$ and $\sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}$, and where ρ is the arithmetical value of $(a^2+b^2)^{\frac{n}{2}}$.

Example. 822. Thus, if $n = \frac{1}{3}$, we find

$$(a + b\sqrt{-1})^{\frac{1}{3}} + (a - b\sqrt{-1})^{\frac{1}{3}} = 2\rho \cos\left(\frac{2r\pi + \theta}{3}\right)$$

where ρ is the arithmetical value of $(a^2 + b^2)^{\frac{1}{6}}$, and r is any term in the series 0, 1, 2, 3, ...

The different values of $2\rho \cos\frac{2r\pi + \theta}{3}$ are

$$2\sqrt[6]{(a^2 + b^2)} \cos\frac{\theta}{3}, \quad -2\sqrt[6]{(a^2 + b^2)} \cos\frac{\pi - \theta}{3},$$

and

$$-2\sqrt[6]{(a^2 + b^2)} \cos\frac{2\pi - \theta}{3}.$$

Thus, if it was required to express the different values of

$$x = \left\{-\frac{r}{2} + \sqrt{\left(\frac{q^3}{27} - \frac{r^2}{4}\right)}\sqrt{-1}\right\}^{\frac{1}{3}} + \left\{-\frac{r}{2} - \sqrt{\left(\frac{q^3}{27} - \frac{r^2}{4}\right)}\sqrt{-1}\right\}^{\frac{1}{3}},*$$

where $\frac{q^3}{27}$ is greater than $\frac{r^2}{4}$, we should find, making

$$\rho = \left(\frac{r^2}{4} + \frac{q^3}{27} - \frac{r^2}{4}\right)^{\frac{1}{6}} = \left(\frac{q^3}{27}\right)^{\frac{1}{6}} = \sqrt{\frac{q}{3}},$$

$$\text{and} \quad \cos\theta = \cos(2r\pi + \theta) = \frac{-\frac{r}{2}}{\sqrt{\frac{q^3}{27}}},$$

$$x = 2\rho \cos\frac{\theta}{3} = 2\sqrt{\frac{q}{3}} \cos\frac{\theta}{3},$$

$$x = 2\rho \cos\frac{2\pi + \theta}{3} = -2\sqrt{\frac{q}{3}} \cos\frac{\pi - \theta}{3},$$

$$x = 2\rho \cos\frac{4\pi + \theta}{3} = -2\sqrt{\frac{q}{3}} \cos\frac{2\pi - \theta}{3}.$$

If more terms in the series are taken, the same values of x recur in the same order.

* This formula will be shewn, in a subsequent Chapter, to express the three roots of the equation

$$x^3 - qx + r = 0,$$

and it will be observed that if $\frac{q^3}{27}$ be less than $\frac{r^2}{4}$, there is no term in the formula which involves $\sqrt{-1}$: or, in other words, $b = 0$.

CHAPTER XXXI.

ON THE REPRESENTATION OF STRAIGHT LINES BOTH IN POSITION AND MAGNITUDE, AND THE APPLICATION OF ALGEBRA TO THE THEORY OF RECTILINEAR FIGURES.

Restatement of the results obtained in the last Chapter respecting the roots of 1.

823. THE investigations, in the last Chapter, have enabled us not merely to prove the necessary existence of roots of the equation

$$x^n - 1 = 0,$$

which are different from 1, {a proposition which we had previously assumed (Art. 708)}, but likewise to determine, in all cases, their complete analytical values: we have there shewn that those roots are expressed by the n values (and there are no more, Art. 811) of the formula

$$\cos\frac{2r\pi}{n} + \sqrt{-1}\,\sin\frac{2r\pi}{n},$$

corresponding to the successive values of r between 0 and $n-1$ inclusive.

The base a corresponding to the least angle of transfer, Art. 729, is determinate.

824. In the application of the successive terms of the period

$$1,\ a,\ a^2, \ldots\ a^{n-1},$$

formed by the n roots of 1, as signs of affection to denote the successive positions of the n radii which divide the circumference of a circle into n equal parts (Art. 728 and 729) it was shewn that if a was the appropriate sign used to denote the least of the successive angles of transfer, then the other terms of the period, in their order, would correspond to the other angles of transfer in their order: but even when the roots of 1 were explicitly given, as in the case of its cubic, biquadratic and quinary roots, we were unable to connect a specific root with a specific angle of transfer, inasmuch as there existed no manifest symbolical connection between this angle and the analytical form of the root which exclusively corresponded to it: this uncertainty however will no longer be found to prevail in the

analytical form of the roots of 1, which Demoivre's formula enables us to assign to them: for, if we make

$$a = \cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}$$

the base of the period

$$1,\ a,\ a^2, \ldots\ a^{n-1},$$

we shall find

$$a^2 = \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)^2 = \cos\frac{4\pi}{n} + \sqrt{-1}\sin\frac{4\pi}{n},$$

$$a^3 = \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)^3 = \cos\frac{6\pi}{n} + \sqrt{-1}\sin\frac{6\pi}{n},$$

$$a^4 = \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)^4 = \cos\frac{8\pi}{n} + \sqrt{-1}\sin\frac{8\pi}{n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a^{n-1} = \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)^{n-1} = \cos\frac{2(n-1)\pi}{n} + \sqrt{-1}\sin\frac{2(n-1)\pi}{n},$$

where the successive angles of transfer $\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots \frac{2r\pi}{n}$ follow the order of the successive powers of a: but if we had assumed

$$a = \cos\frac{4\pi}{n} + \sqrt{-1}\sin\frac{4\pi}{n}$$

as the base of the period, we should have found

$$a^2 = \cos\frac{8\pi}{n} + \sqrt{-1}\sin\frac{8\pi}{n},$$

$$a^3 = \cos\frac{12\pi}{n} + \sqrt{-1}\sin\frac{12\pi}{n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a^{n-1} = \cos\frac{4(n-1)\pi}{n} + \sqrt{-1}\sin\frac{4(n-1)\pi}{n},$$

where the order of the powers of the base of the period does not follow the order of the angles, and where we must pass twice round the circumference of the circle before we return to the

primitive line: in a similar manner, if the base of the period had been assumed to be

$$\cos\frac{2r\pi}{n}+\sqrt{-1}\,\sin\frac{2r\pi}{n},$$

we should have returned to the primitive line after r transits round the circumference, and not before. It appears, therefore, that the base a, which corresponds to the least angle of transfer is *determinate*, and is in all cases that root of 1, which is expressed by the formula

$$\cos\frac{2\pi}{n}+\sqrt{-1}\,\sin\frac{2\pi}{n},$$

where $\frac{2\pi}{n}$ is the angle of transfer, and n is the denomination of the root.

825. It follows, therefore, in conformity with the conclusions established in Art. 728, that if AB or ρ be the primitive line, and if AB, AB_1, AB_2, ... AB_{n-1} be drawn from the centre of the circle to a series of points dividing the circumference into n equal parts, and making angles equal to $\frac{2\pi}{n}$ with each other, then the several radii thus drawn will be represented both in magnitude and in position, in their order, by

Resumed consideration of the interpretations given in Art. 728.

(1) ρ,

(2) $\rho\left(\cos\frac{2\pi}{n}+\sqrt{-1}\,\sin\frac{2\pi}{n}\right)$,

(3) $\rho\left(\cos\frac{4\pi}{n}+\sqrt{-1}\,\sin\frac{4\pi}{n}\right)$,

....

(n) $\rho\left\{\cos\frac{2(n-1)\pi}{n}+\sqrt{-1}\,\sin\frac{2(n-1)\pi}{n}\right\}$;

and that if the formation of these analytical values, according to the same law, be continued, the same series of values will be reproduced in the same order: for

$$\rho\left(\cos\frac{2n\pi}{n}+\sqrt{-1}\,\sin\frac{2n\pi}{n}\right)=\rho,$$

$$\rho\left\{\cos\frac{2(n+1)\pi}{n}+\sqrt{-1}\,\sin\frac{2(n+1)\pi}{n}\right\}=\rho\left(\cos\frac{2\pi}{n}+\sqrt{-1}\,\sin\frac{2\pi}{n}\right),$$

....

$$\rho\left(\cos\frac{2(n+r)\pi}{n}+\sqrt{-1}\sin\frac{2(n+r)\pi}{n}\right)$$
$$=\rho\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right);$$

. . . .

and, consequently, the n roots of 1, connected as signs of affection with a symbol ρ which denotes a line, will represent the same line both in magnitude and position in n different positions, making angles equal to $\frac{2\pi}{n}$ with each other, and *in no more;* the successive lines, in their order, being severally represented by the successive values of

$$\rho\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)$$

where r has successively every value in the series

$$0,\ 1,\ 2,\ \ldots\ (n-1).$$

The same interpretations generalized.

826. And, generally, if ρ be the length of a line, making an angle θ, with the primitive line, then

$$(\cos\theta+\sqrt{-1}\sin\theta)\,\rho$$

will, in conformity with the preceding theory, express it both in magnitude and in position: for if $\theta=\frac{2r\pi}{n}$ where the values of r and n are whole numbers, then

$$\cos\theta+\sqrt{-1}\sin\theta$$

is one of the n roots of 1, and therefore

$$\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\rho,$$

or the equivalent expression

$$(\cos\theta+\sqrt{-1}\sin\theta)\,\rho$$

will represent the magnitude and position of a line, equal in magnitude to ρ, which makes an angle $\frac{2r\pi}{n}$ or θ with the primitive line: and if no finite integral values of r and n can be found, which make $\frac{2r\pi}{n}$ *absolutely* equal to θ, yet we can always determine, by the theory of converging fractions or otherwise,

such values of them as will make the value of $\frac{2r\pi}{n}$ approximate to θ as near as we chuse: it will follow, therefore, under such circumstances, that

$$\left(\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}\right)\rho$$

will represent a line in magnitude and position (Art. 820) as near as we chuse to that which is assumed to be expressed by

$$(\cos\theta + \sqrt{-1}\sin\theta)\,\rho:$$

and if the approximation be indefinitely continued, we may consider the line which is represented by one of these formulæ as *geometrically coincident* (Art. 169) with that which is assumed to be represented by the other.

Thus, if $\theta = 13^\circ.\,14'$, and if we make $\frac{2r\pi}{n} = \theta$, we shall find the series of fractions Example.

$$\frac{1}{27},\ \frac{4}{109},\ \frac{5}{136},\ \frac{49}{1333},\ \frac{397}{10800}$$

converging to the value of $\frac{r}{n}$, the last of them being equal to it: of these, the second fraction gives $\frac{2r\pi}{n} = \frac{8\pi}{109} = 13^\circ.\,12'.66$, which is less than θ: the third gives $\frac{2r\pi}{n} = \frac{10\pi}{136} = 13^\circ.\,14'.\,1$, which is greater than θ: the fourth gives $\frac{2r\pi}{n} = \frac{98\pi}{1333} = 13^\circ.\,13'.998$, which differs from θ by less than $\frac{1}{500}$th part of a minute: the last, which gives the accurate value of θ, is the line expressed in magnitude and position by

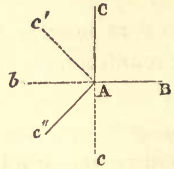
$$\left(\cos\frac{794\pi}{10800} + \sqrt{-1}\sin\frac{794\pi}{10800}\right)\rho;$$

whose sign of affection is the 397th term of the period formed by the 10800 values of the 10800th root of 1.

827. Inasmuch, therefore, as a line ρ, making an angle θ with a primitive line AB, is expressed, both in magnitude and position with respect to it, by Examples of the interpretations of $\cos\theta + \sqrt{-1}\sin\theta$ for given values of θ.

$$(\cos\theta + \sqrt{-1}\sin\theta)\,\rho,$$

it follows, that if $\theta = 90^\circ$, or if AC be perpendicular to AB, then AC is expressed by $\rho\sqrt{-1}$ (Art. 733): if θ be 120°, as in the position AC', then AC' is expressed by



$$\left(-\frac{1}{2} + \frac{\sqrt{3}\sqrt{-1}}{2}\right)\rho \text{ (Art. 734):}$$

if $\theta = 180^\circ$, as in the position Ab, opposite to AB, then Ab is expressed by $(-1)\rho$ or $-\rho$ (Arts. 732 and 559): if $\theta = 240^\circ$ as in the position AC'', then AC'' is expressed by

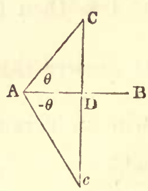
$$\left(-\frac{1}{2} - \frac{\sqrt{3}\sqrt{-1}}{2}\right)\rho \text{ (Art. 733):}$$

if $\theta = 270^\circ$, as in the position Ac, then Ac is expressed by $-\rho\sqrt{-1}$ (Art. 733) where Ac, being opposite to AC, is distinguished, in its representation, from it by the additional sign $-$: the preceding results are in accordance with those established in Chap. XXV, and are sufficient to shew the ease and readiness with which this principle may be applied.

If two sides of a right-angled triangle be expressed in magnitude and position by a and $b\sqrt{-1}$, its hypothenuse will be expressed by $a+b\sqrt{-1}$.

828. Again, since

$$\rho(\cos\theta + \sqrt{-1}\sin\theta) = \rho\cos\theta + \rho\sin\theta\sqrt{-1},$$



it will appear that if, upon the primitive line AB, we take $AD = \rho\cos\theta$ and DC equal in magnitude to $\rho\sin\theta$ and perpendicular to AB, then the hypothenuse AC of the triangle ADC is equal to ρ, and makes an angle θ with AB, and is therefore represented in magnitude and in position by $\rho\cos\theta + \rho\sin\theta\sqrt{-1}$.

And if we further replace $\rho\cos\theta$ by a and $\rho\sin\theta$ by b, then a and $b\sqrt{-1}$ express the two sides AD and DC of the right-angled triangle ADC both in magnitude and position (Art. 827), and $a+b\sqrt{-1}$ likewise expresses the magnitude and position of its hypothenuse: and in a similar manner, if we replace θ by $-\theta$, and therefore $\sin\theta$ by $-\sin\theta$, and if, upon the primitive line AB, we take $AD = \rho\cos\theta$ and express Dc (equal and opposite to DC) by $-\rho\sin\theta$, then the hypothenuse Ac of the right-angled triangle

ADc will be correctly expressed both in magnitude and position by

$$\rho(\cos - \theta + \sqrt{-1}\sin - \theta) \text{ or } \rho(\cos\theta - \sqrt{-1}\sin\theta):$$

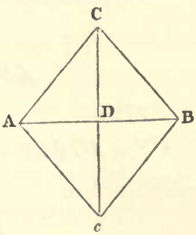
for Ac is equal in magnitude to ρ, and makes an angle $-\theta$ with the primitive line: in this case, therefore, a and $-b\sqrt{-1}$ will express the magnitudes and positions of the two sides of the right-angled triangle ADc, and $a - b\sqrt{-1}$ will express the magnitude and position of its hypothenuse: it follows therefore, generally, that the *symbolical sum* of two lines AD and DC which are represented in magnitude and position by a and $b\sqrt{-1}$, will be equal to the hypothenuse of the right-angled triangle which they form, whose magnitude is expressed by $\sqrt{(a^2+b^2)}$ and which makes an angle with the primitive line whose cosine is

$$\frac{a}{\sqrt{(a^2+b^2)}} \text{ and sine } \frac{b}{\sqrt{(a^2+b^2)}}:$$

and also that the *symbolical difference* of two lines AD and DC, or $a - b\sqrt{-1}$, will be the hypothenuse of the right-angled triangle formed by AD and Dc, where Dc is equal and opposite to DC, and where Ac is equal in magnitude to $\sqrt{(a^2+b^2)}$ and makes an angle with the primitive line or a whose cosine is $\frac{a}{\sqrt{(a^2+b^2)}}$ and whose sine is $-\frac{b}{\sqrt{(a^2+b^2)}}$.

829. It appears, from the last Article, that if AC and $Ac\,(\rho)$ be equal to each other, and make the angles θ and $-\theta$ respectively with the primitive line AB, they are severally represented in magnitude and position by

The sum or difference of two equal lines considered in position as well as magnitude are the diagonals of the rhombus which they form.



$$\rho(\cos\theta + \sqrt{-1}\sin\theta)$$

and

$$\rho(\cos\theta - \sqrt{-1}\sin\theta).$$

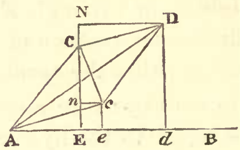
If we add and subtract these expressions for AC and Ac, we get, for their *symbolical sum*, $2\rho\cos\theta$, and for their *symbolical difference* $2\rho\sin\theta\sqrt{-1}$: but if the rhombus $ACBc$ be completed, and its diagonals AB and Cc be drawn, then we find

$$AB = 2AD = 2AC\cos\theta \text{ and } Cc = 2CD = 2AC\sin\theta:$$

and inasmuch as Cc is perpendicular to the primitive line AB, it is expressed both in magnitude and position by $2\rho \sin\theta \sqrt{-1}$: of the diagonals of a rhombus, therefore, the one is the *symbolical sum* of the two sides which include it, and the other is their *symbolical difference.*

The sum of two adjacent sides of a parallelogram, considered in position as well as magnitude, is the diagonal which they include: their difference is the other diagonal.

830. More generally, if $AC(\rho)$ and $Ac(\rho')$ be two lines making angles $CAB(\theta)$ and $cAB(\theta')$ with the primitive line AB respectively, then they are represented in magnitude and position by



$$\rho(\cos\theta + \sqrt{-1}\sin\theta)$$

and

$$\rho'(\cos\theta' + \sqrt{-1}\sin\theta')$$

respectively: and if we add and subtract these expressions we shall find for their *symbolical sum*

$$\rho\cos\theta + \rho'\cos\theta' + (\rho\sin\theta + \rho'\sin\theta')\sqrt{-1},$$

and for their *symbolical difference*

$$\rho\cos\theta - \rho'\cos\theta' + (\rho\sin\theta - \rho'\sin\theta')\sqrt{-1}:$$

if we now complete the parallelogram $ACDc$ contained by AC and Ac, and draw CE, ce and Dd perpendicular to the primitive line AB, and DN and cn perpendicular to CE or CE produced either way, we shall find

$$AE = AC\cos\theta = \rho\cos\theta, \text{ and } Ae = DN = Ed = \rho'\cos\theta':$$

and therefore

$$Ad = AE + Ae = \rho\cos\theta + \rho'\cos\theta'.$$

Again,

$$CE = AC\sin\theta = \rho\sin\theta, \text{ and } ce = CN = Ac\sin\theta' = \rho'\sin\theta';$$

and therefore

$$NE = Dd = \rho\sin\theta + \rho'\sin\theta'.$$

It follows, therefore, that Ad and Dd are represented in magnitude and in position by

$$\rho\cos\theta + \rho'\cos\theta', \text{ and } (\rho\sin\theta + \rho'\sin\theta')\sqrt{-1}$$

respectively, and therefore their *symbolical sum* is the hypothenuse AD of the right-angled triangle which they form.

Again, we similarly find

$$-Ee = -cn = \rho \cos \theta - \rho' \cos \theta':$$

and

$$Cn = \rho \sin \theta - \rho' \sin \theta':$$

and since cn, which is parallel to the primitive line AB, is represented in magnitude and in position by $\rho \cos \theta - \rho' \cos \theta'$ and Cn, which is perpendicular to AB, is represented in magnitude and position by $(\rho \sin \theta - \rho' \sin \theta') \sqrt{-1}$, it follows that Cc, which is the hypothenuse of the right-angled triangle which they form, and which is that diagonal of the parallelogram contained by AC and Ac which is not included by them, is the *symbolical difference* of AC and Ac.

831. Inasmuch as the diagonal AD is expressed in position and magnitude by Expressions for the diagonal, in magnitude and position.

$$\rho \cos \theta + \rho' \cos \theta' + (\rho \sin \theta + \rho' \sin \theta') \sqrt{-1},$$

and inasmuch, as if d was its length and ϕ the angle which it made with the primitive line, it would equally be expressed by

$$d(\cos \phi + \sqrt{-1} \sin \phi),$$

it follows that (Art. 828)

$$d = \sqrt{\{(\rho \cos \theta + \rho' \cos \theta')^2 + (\rho \sin \theta + \rho' \sin \theta')^2\}}$$
$$= \sqrt{\{\rho^2 + \rho'^2 + 2\rho\rho' \cos(\theta - \theta')\}},$$

and $\cos \phi = \dfrac{\rho \cos \theta + \rho' \cos \theta'}{d}$.

The corresponding expressions for the second diagonal Cc or d', and the angle ϕ' which it makes with the primitive line, would be

$$d' = \sqrt{\{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta + \theta')\}},$$

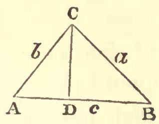
and $\cos \phi' = \dfrac{\rho \cos \theta - \rho' \cos \theta'}{d'}$.

832. The interpretations which form the subjects of the preceding articles, enable us to represent lines generally in position as well as in magnitude, and thus to bring their properties under the dominion of Algebra: it may tend to illustrate the theory of such interpretations, as well as the relations of Algebra to Geometry, if we proceed to apply them to some of the more common properties of geometrical figures which are immediately deducible from them. Application of these interpretations to the theory of rectilinear figures.

Symbolical representation of the three sides of a triangle.

833. Let it be required to express symbolically the three sides AB, BC, and CA of a triangle, taken in order.

Let the sides of the triangle ABC be expressed in magnitude by a, b, c, and the angles opposite to them by A, B, and C: and let c be the primitive line: then BC or a, which makes an angle $\pi - B$ with c, will be expressed by



$$a\{\cos(\pi - B) + \sqrt{-1}\,\sin(\pi - B)\},$$

$$\text{or}\quad a(-\cos B + \sqrt{-1}\,\sin B)^*;$$

and AC or b, which makes an angle with c equal to $2\pi - (B + C)$ or $\pi + A$, will be expressed by

$$b(-\cos A - \sqrt{-1}\,\sin A).$$

The sum of the three sides of a triangle, taken in order, is zero.

834. The sum of the three sides of a triangle taken in order is equal to zero.

For, if CD be drawn perpendicular to AB, we shall find $a\sin B = CD = b\sin A$: and $AB = AD + DB = b\cos A + a\cos B = c$: and it follows therefore that the sum of the three sides of the triangle ABC, taken in order, is equal to

$$c + a(-\cos B + \sqrt{-1}\,\sin B) + b(-\cos A - \sqrt{-1}\,\sin A)$$
$$= c - a\cos B - b\cos A + \sqrt{-1}\,(a\sin B - b\sin A)$$
$$= 0.\dagger$$

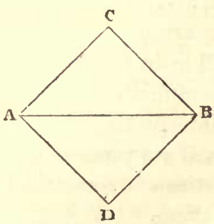
The same conclusion is expressed by saying that the symbolical sum of two sides AC and CB of a triangle ACB, taken in order, is equal to the third side AB, estimated in the direction AB, and not in the reversed direction BA.

Fundamental equations in Trigonometry properly so called.

It will be shewn, in a subsequent Chapter, that all the relations of the sides and angles of triangles, upon which their

* For the exterior angle at B is $\pi - B$, and at C is $\pi - C$: and the entire angle of transfer, in passing from the position AB to CA, is $2\pi - B - C = \pi + A$: for $A + B + C = \pi$.

† The same conclusion follows from the proposition proved in Art. 829: for if we complete the parallelogram $ACBD$, then the symbolical sum of AC and AD is AB: and since AD is equal to CB and estimated in the same direction with it, it is symbolically identical with CB: it follows therefore that the symbolical sum of AC and CB is AB.

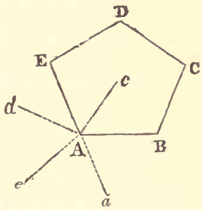


solution or determination, from the requisite data, will depend, and which constitute the *proper* science of Trigonometry, are deducible from the three equations

$$\left.\begin{array}{r} c = a \cos B + b \cos A \\ a \sin B = b \sin A \\ A + B + C = \pi \end{array}\right\}.$$

835. It will follow, as an immediate consequence of the proposition in the last Article, that the sum of the sides of any rectilineal figure, taken in order, when estimated both in position and magnitude, is equal to zero: for if *ABCDE* be any rectilineal figure, then the sum of the consecutive sides *AB* and *BC* is the line formed by joining *AC*: the sum of *AC* and *CD*, or of *AB*, *BC*, and *CD* is the line formed by joining *AD*: the sum of *AD* and *DE*, or of *AB*, *BC*, *CD* and *DE* is the last side of the figure *AE*: and, finally, the sum of *EA* and *AE* or of *AB*, *BC*, *CD*, *DE*, and *EA* is zero, inasmuch as *AE* and *EA* have opposite signs: the same reasoning will apply, whatever be the number of sides.

The proposition in Art. 834, extended to any rectilineal figure.



836. It follows, therefore, that if we draw *Ac*, *Ad*, *Ae*, and *Aa* parallel to *BC*, *CD*, *DE* and in the direction of *EA* produced respectively, and if θ, θ', θ'', θ''' be the goniometrical angles which *Ac*, *Ad*, *Ae*, and *Aa* make with the primitive line *AB*, and if we represent the lengths of the sides *AB*, *BC*, *CD*, *DE* and *EA* by *a*, *b*, *c*, *d*, and *e*, then the symbolical sum of the sides will be

General equation of figure.

$$a + b(\cos\theta + \sqrt{-1}\sin\theta) + c(\cos\theta' + \sqrt{-1}\sin\theta')$$
$$+ d(\cos\theta'' + \sqrt{-1}\sin\theta'') + e(\cos\theta''' + \sqrt{-1}\sin\theta''') = 0,$$

which becomes, by collecting together those terms which are affected, and those which are not affected, by the sign $\sqrt{-1}$,

$$a + b\cos\theta + c\cos\theta' + d\cos\theta'' + e\cos\theta'''$$
$$+ (b\sin\theta + c\sin\theta' + d\sin\theta'' + e\sin\theta''')\sqrt{-1} = 0,$$

an equation which is equivalent to the two equations*

* For if $A + B\sqrt{-1} = 0$, then $A = 0$ and $B = 0$: for we get $A = -B\sqrt{-1}$, and therefore $A^2 = (-1)^2 . B^2 . (-1) = -B^2$, a condition which no arithmetical values of A and B or of $-A$ or $-B$, which are different from *zero*, can satisfy.

$$a + b\cos\theta + c\cos\theta' + d\cos\theta'' + e\cos\theta''' = 0 \qquad (a),$$
$$b\sin\theta + c\sin\theta' + d\sin\theta'' + e\sin\theta''' = 0 \qquad (b).$$

Modifications which the equations of figure receive from the equation of angles.

837. If A, B, C, D, E be the interior, and A', B', C', D', E' the corresponding exterior angles of the figure, and therefore, (Euclid, Book I. Prop. 32, Cor. 1 and 2),

$$A' + B' + C' + D' + E' = 2\pi,$$
$$A + B + C + D + E = 3\pi \qquad (c),$$

we get

$$\theta = B' = \pi - B,$$
$$\theta' = B' + C' = 2\pi - (B + C),$$
$$\theta'' = B' + C' + D' = 3\pi - (B + C + D),$$
$$\theta''' = B' + C' + D' + E' = 4\pi - (B + C + D + E);$$

if we now replace θ, θ', θ'', θ''' in equations (a) and (b), by these values, they will assume the form

$$a - b\cos B + c\cos(B+C) - d\cos(B+C+D) + e\cos(B+C+D+E) = 0,$$
$$b\sin B - c\sin(B+C) + d\sin(B+C+D) - e\sin(B+C+D+E) = 0.$$

More generally, if the rectilineal figure have n sides $a, a_1, a_2, \ldots a_{n-1}$, and if its interior angles be $A, A_1, A_2, \ldots A_{n-1}$ (where A is the angle at the point from which a is reckoned), then the equations (a), (b), and (c) will become *

$$a - a_1\cos A_1 + a_2\cos(A_1 + A_2) - \ldots + (-1)^{n-1} a_{n-1}\cos(A_1 + A_2 + \ldots A_{n-1}) = 0 \ (a),$$
$$a_1\sin A_1 - a_2\sin(A_1 + A_2) + \ldots + (-1)^n a_{n-1}\sin(A_1 + A_2 + \ldots A_{n-1}) = 0 \ (b),$$
$$A + A_1 + A_2 + \ldots\ldots\ldots\ldots\ldots A_{n-1} = (n-2)\pi \qquad (c).$$

Again, since

$$A_1 + A_2 + \ldots A_{n-1} = (n-2)\pi - A,$$
$$A_1 + A_2 + \ldots A_{n-2} = (n-2)\pi - (A + A_{n-1}),$$
$$A_1 + A_2 + \ldots A_{n-3} = (n-2)\pi - (A + A_{n-2} + A_{n-1}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

* We write the last term $+(-1)^{n-1} a_{n-1} \cos(A_1 + A_2 + \ldots A_{n-1})$ in order to intimate that it is preceded by a positive or a negative sign, according as n is *odd* or *even*: for in the first case $(-1)^{n-1} = +1$, and in the second $(-1)^{n-1} = -1$.

the equations (a) and (b) may be put under the form

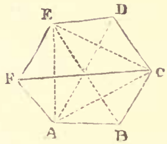
$$a - a_1 \cos A_1 + a_2 \cos (A_1 + A_2) - \ldots + (-1)^{n-2} a_{n-2} \cos (A + A_{n-1})$$
$$+ (-1)^{n-1} a_{n-1} \cos A = 0 \qquad (a),$$

$$a_1 \sin A_1 - a_2 \sin (A_1 + A_2) + \ldots + (-1)^{n-1} a_{n-2} \sin (A + A_{n-1})$$
$$+ (-1)^n a_{n-1} \sin A = 0 \qquad (b).$$

The equations (a) and (b) may be called *equations of figure*, inasmuch as they are not satisfied unless the lines, from whose symbolical sum these equations are derived, either form a figure, or are capable of forming one, when arranged in consecutive order: if their sum is not equal to zero, but to $\alpha + \beta\sqrt{-1}$, then $\alpha + \beta\sqrt{-1}$ will express the line, in position and magnitude, which completes the figure. Equations of figure.

838. Thus, if a be the side AB of a regular hexagon $ABCDEF$, then its several sides are represented by a, $a(\cos 60^0 + \sqrt{-1}\sin 60^0)$, Expression of the diagonals of a regular hexagon.

$$a(\cos 120^0 + \sqrt{-1}\sin 120^0),$$
$$-a,\ -a(\cos 60^0 + \sqrt{-1}\sin 60^0),$$
$$-a(\cos 120^0 + \sqrt{-1}\sin 120^0),$$



since the three last sides are parallel to the three first, but estimated in opposite directions.

The sum of AB and BC or

$$AC = a + a(\cos 60^0 + \sqrt{-1}\sin 60^0) = \frac{3a}{2} + \frac{a\sqrt{3}}{2}\sqrt{-1},$$

a line whose length is $\sqrt{\left(\frac{9a^2}{4} + \frac{3a^2}{4}\right)} = a\sqrt{3}$, and which makes an angle with AB whose cosine is $\frac{\sqrt{3}}{2}$ and sine $\frac{1}{2}$, and is therefore 30^0.

The sum of AB, BC and CD or

$$AD = a + a(\cos 60^0 + \sqrt{-1}\sin 60^0) + a(\cos 120^0 + \sqrt{-1}\sin 120^0)$$
$$= a + a\sqrt{3}\sqrt{-1},$$

which is a line whose length is $2a$, making an angle with AB

whose cosine is $\frac{1}{2}$ and sine $\frac{\sqrt{3}}{2}$, and therefore of 60^0: this line is a diameter of the circumscribing circle.

The sum of AB, BC, CD and DE or

$$AE = a + a(\cos 60^0 + \sqrt{-1}\sin 60^0) + a(\cos 120^0 + \sqrt{-1}\sin 120^0) - a$$
$$= a\sqrt{3}\sqrt{-1},$$

which is a line whose length is $a\sqrt{3}$, and at right angles to AB.

The sum of AB, BC, CD, DE and EF or AF

$$= a + a(\cos 60^0 + \sqrt{-1}\sin 60^0) + a(\cos 120^0 + \sqrt{-1}\sin 120^0)$$
$$- a - a(\cos 60^0 + \sqrt{-1}\sin 60^0) = a(\cos 120^0 + \sqrt{-1}\sin 120^0).$$

In a similar manner, it may be shewn that the diagonal CE is the sum of CD and DE, or it is the difference of the sums of AB, BC, CD, and DE, and of AB and BC, or it is the difference of AE and AC: in whatever manner it may be deduced, it will be found to be represented by

$$-\frac{3a}{2} + \frac{a\sqrt{3}\sqrt{-1}}{2},$$

which is equivalent to

$$a\sqrt{3}(\cos 120^0 + \sqrt{-1}\sin 120^0),$$

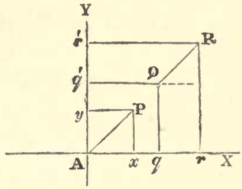
which is a line whose length is $a\sqrt{3}$, and which makes an angle of 120^0 with the primitive line AB.

It is obvious that the same principles may be applied to express, both in magnitude and in direction, the diagonals of any rectilineal figure whatsoever.

The orthographical projections of lines upon rectangular axes.

839. In the preceding Articles we have considered lines in their *relative*, and not in their *absolute*, positions, no distinction being made in the representation of lines which are equal and parallel to each other and also estimated in the same direction: but if the primitive line be extended both ways, and another line be drawn at right angles to it through its beginning or origin, then such lines are called *rectangular axes*, and lines may always be expressed in magnitude and in relative position,

when their *orthographical* projections upon such *axes* are given: thus, if AP be a line drawn from the origin or intersection of the axes, and if Px and Py be drawn perpendicular to AX and AY respectively, then Ax and Ay are the *orthographical projections* of AP: and if QR be a line equal and parallel to AP and estimated in the same direction with it, and if Qq and Rr, Qq' and Rr' be drawn perpendicular to AX and AY respectively, then qr and $q'r'$ are its *orthographical projections*, which are also respectively equal to those of AP.



840. The orthographical projections of a line which is represented in magnitude and direction by

A line is given in magnitude and relative position when its orthographical projections are given.

$$a(\cos\theta + \sqrt{-1}\sin\theta)$$

are $a\cos\theta$ upon the primitive line, and $a\sin\theta$ upon the line which is perpendicular to it: and conversely, if α and β be the orthographical projections of a line upon the primitive line and upon an axis perpendicular to it, then the line itself will be expressed in magnitude and position by

$$\alpha + \beta\sqrt{-1}$$

or by the equivalent expression

$$\sqrt{(\alpha^2+\beta^2)}\,(\cos\phi + \sqrt{-1}\sin\phi),$$

$$\text{where } \cos\phi = \frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}} \text{ and } \sin\phi = \frac{\beta}{\sqrt{(\alpha^2+\beta^2)}},$$

$\sqrt{(\alpha^2+\beta^2)}$ being the length of the line, and ϕ the angle which it makes with the primitive line.

841. Again, the perpendiculars let fall from a point P upon the two axes AX and AY at right angles to each other, are also called its *co-ordinates*, and they are likewise equal to the orthographical projections of the line AP, which is drawn from the origin A to P, to which they are respectively parallel: namely, the co-ordinate Px is equal to Ay and the co-ordinate Py to Ax: and if we take the co-ordinates of the points Q and R, at the extremities of the line QR which is equal and parallel to, and symbolically identical with, AP, then the difference qr of the co-ordinates which are parallel to AX will be equal to the

Relations of rectangular co-ordinates and orthographical projections.

orthographical projection of QR upon AX, and the difference $q'r'$ of the co-ordinates which are parallel to AY will be equal to the orthographical projection of QR upon AY.

The theory of co-ordinates is the basis of the application of Algebra to Geometry, where absolute as well as relative position is considered.

The theory of co-ordinates will be found to form the basis of the application of Algebra to Geometry, where lines are considered in their *absolute* as well as in their *relative* positions: it is a theory full of important consequences and which will require very extensive developement, and we shall therefore reserve the further consideration of it to a subsequent part of this work.

Conditions of the movement of a point which perpetually describes the same closed figure.

842. If we should conceive a point to be moved through spaces or lines represented in magnitude by

$$a, \ a_1, \ a_2, \dots a_{n-1},$$

and in directions making the goniometrical angles

$$\theta, \ \theta_1, \ \theta_2, \dots \theta_{n-1},$$

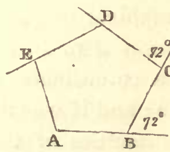
with the primitive line, then the point would return to the origin of the movement, and the lines described would form a *closed* figure, when the equations in Art. 837, were verified: for if the symbolical sums of the orthographical projections upon the two rectangular axes are equal to zero, the line joining the origin and the final point would be equal to zero likewise: and it would further follow, that if the sum of the angles of transfer or

$$A' + A'_1 + A'_2 + \dots A'_{n-1},$$

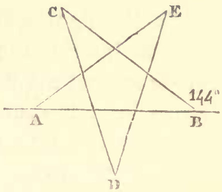
including the angle of transition from the n^{th} to the $(n+1)^{\text{th}}$ line or from a_{n-1} to a was 2π or *any multiple of* 2π, and if the conditions of the movement remained unaltered, the point would continue to circulate in the same figure for ever: we will endeavour to illustrate our meaning, and to demonstrate the conclusion to which it leads, in the case of the movement of a point which describes a regular pentagon.

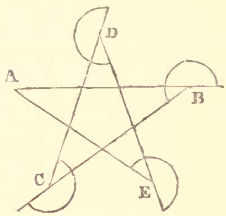
In a case of a point describing a regular pentagon, whether geometrical or stellated,

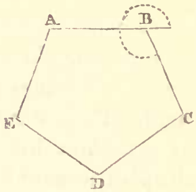
843. Thus, if the successive spaces or lines described be equal to each other (a), and if, at the end of each line, the direction of the movement is *instantaneously* changed through an angle of transfer of 72°, then the point will return into itself after describing five such lines: and if the movement be further continued in conformity with the same conditions, the point will for ever circulate in the same figure: but if, all the other



assumptions remaining, we suppose the angle of transfer, at the end of each line to be 144^0 instead of 72^0 the point would still return into itself after describing five lines, but the lines would intersect each other, and the figure would be *stellated:* thus, if AB was the primitive line, the order of description of the lines would be AB, BC, CD, DE, and EA, the angles at A, B, C, D, E, being severally 36^0 or the supplements of 144^0: again, if the angle of transfer at the end of each line was 3×72^0 or 216^0, the same stellated pentagonal figure would be described, but in a reversed position: and if the angle of transfer was 4×72^0 or 288^0, the ordinary pentagon of geometry would be described, but on a different side of the primitive line AB, to that on which the pentagon corresponding to an angle of transfer of 72^0 was described: but if the angle of transfer was 360^0, the point would continue to move forward in the same direction and in the same straight line for ever.







When the movement continues in a straight line.

844. If we should define a regular pentagon to be a figure of five equal sides and five equal angles, without any assumed exclusion of *re-entrant* angles or intersecting sides, it is obvious that the *stellated* figures referred to in the last Article would equally answer the conditions of such a definition with the regular and completely bounded pentagon of Geometry: the essential distinction between them presents itself in the equation of angles, which is commonly assumed in Geometry to be *unique* for all figures of the same number of sides: thus the equations of angles for the different angles of transfer which are considered in the last Article are, if A, B, C, D, E be the interior angles,

Theory of periodical movements corresponding to different equations of angles.

$$(1) \quad A + B + C + D + E = 3\pi,$$
$$(2) \quad A + B + C + D + E = \pi,$$
$$(3) \quad A + B + C + D + E = -\pi,$$

$$(4)\quad A+B+C+D+E=-3\pi.$$

$$(5)\quad A+B+C+D+E=-5\pi.$$

In the case of pentagonal movements.

Thus, in the figures corresponding to the equations (1) and (4), the several interior angles are equal to each other, but with different signs, and the figures are formed on different sides of the primitive line: the same remark applies to the stellated figures which correspond to the equations (2) and (3): in Geometry these distinctions of position, as above and below the primitive line, are not recognized, and our attention therefore in that science is necessarily confined to one figure of each class: the fifth equation intimates, that as each interior angle is $-\pi$, and as the sum of each exterior and interior angle is π, each exterior angle is 2π, and consequently no figure is formed.

In the case of hexagonal movements.

845. The angle of transfer in a regular hexagon is $\frac{\pi}{3}$ or 60°: if we should suppose the sum of the exterior angles doubled, the angle of transfer would be doubled or become 120°, in which case the corresponding hexagon would degenerate into an equilateral or *regular* triangle and the describing point would pass twice over each of its sides in six changes of movement: if we supposed the angle of transfer tripled, the figure corresponding would become a straight line described by an oscillating movement: if quadrupled it would become the regular hexagon of Geometry, in a reversed position to the former, in which each of the interior angles was negative and equal to $\frac{-\pi}{3}$: if quintupled, it would again become an equilateral triangle in a reversed position to the former: if sextupled, the point would continue to move forward in the same straight line, in the direction of its primitive motion: if still higher multiples of the angle of transfer were taken, the same series of movements would be reproduced, and in the same order.

In the case of heptagonal and octagonal movements.

The series of movements corresponding to an angle of transfer of $\frac{2\pi}{7}$ and of its successive multiples, would be distributed into successive periods of seven figures or lines, of which the 1st and 6th would be the regular heptagons of Geometry in reversed positions, the 2nd and 5th, 3rd and 4th stellated figures in reversed positions, and the 7th a continued and inde-

finite straight line: in the series of movements corresponding to an angle of transfer of $\frac{2\pi}{8}$ and of its successive multiples, the 1st and 7th would be the regular octagons of Geometry in reversed positions, the 2nd and 6th squares in reversed positions, the 3rd and 5th octangular stellated figures, the 4th a terminated and the 8th an indefinite, straight line. It is not necessary to extend these observations further, as may be easily done, to the series of movements corresponding to other angles of transfer and their successive multiples.*

* The general theory of such periodic movements requires that the sums of the orthographical projections of the n first lines described should be equal to zero, where $\frac{2\pi}{n}$, or any of its successive multiples, is the least angle of transfer: or if we express this angle of transfer by θ, then the equations of the figures described or of the movement of a point, which, after describing n equal spaces, returns into itself, are

$$\rho\,\{1 + \cos\theta + \cos 2\theta + \dots\dots\dots \cos(n-1)\theta\} = 0, \qquad (s)$$

$$\rho\,\{\sin\theta + \sin 2\theta + \dots\dots\dots \sin(n-1)\theta\} = 0. \qquad (s')$$

If we replace $\cos\theta$ by $\frac{a^{\theta} + a^{-\theta}}{2}$ (Art. 807), the first series (s) becomes

$$\rho\left(1 + \frac{a^{\theta} + a^{-\theta}}{2} + \frac{a^{2\theta} + a^{-2\theta}}{2} + \dots\dots\dots \frac{a^{(n-1)\theta} + a^{-(n-1)\theta}}{2}\right)$$

$$= \frac{\rho}{2}\left(1 + a^{\theta} + a^{2\theta} + \dots\dots\dots a^{(n-1)\theta}\right) + \frac{\rho}{2}\left(1 + a^{-\theta} + a^{-2\theta} + \dots\dots\dots a^{-(n-1)\theta}\right)$$

$$= \frac{\rho}{2}\left(\frac{a^{n\theta} - 1}{a^{\theta} - 1} + \frac{a^{-n\theta} - 1}{a^{-\theta} - 1}\right) \quad \text{(Art. 429)}.$$

If we divide the numerator and denominator of the first and second of the fractions of which this expression is composed by $a^{\frac{\theta}{2}}$ and $a^{-\frac{\theta}{2}}$ respectively, we get

$$s = \frac{\rho}{2}\left(\frac{a^{(n-\frac{1}{2})\theta} - a^{-(n-\frac{1}{2})\theta} + a^{\frac{\theta}{2}} - a^{-\frac{\theta}{2}}}{a^{\frac{\theta}{2}} - a^{-\frac{\theta}{2}}}\right),$$

which becomes, if we replace $a^{(n-\frac{1}{2})\theta} - a^{-(n-\frac{1}{2})\theta}$ by $2\sqrt{-1}\sin(n-\frac{1}{2})\theta$ and $a^{\frac{\theta}{2}} - a^{-\frac{\theta}{2}}$ by $2\sqrt{-1}\sin\frac{\theta}{2}$,

$$s = \frac{\rho}{2}\left\{\frac{\sin(n-\frac{1}{2})\theta + \sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right\} = \frac{\rho\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}} \quad \text{(Art. 773)}.$$

In a

Number of independent elements which are assumable in a figure of n sides.

846. In a figure of n sides, there are $2n$ elements, and only three necessary equations amongst them, one of which is the equation of angles: there are, therefore, $2n-3$ elements which are indeterminate and assumable at pleasure, though not absolutely unlimited in themselves,* which are n sides and $(n-3)$

In a similar manner, if s' be the sum of the series

$$\rho\,(\sin\theta + \sin 2\theta + \text{.........} \sin n\theta),$$

we shall find

$$s' = \frac{\rho \sin\frac{n\theta}{2} \sin\left(\frac{n-1}{2}\right)\theta}{\sin\frac{\theta}{2}}.$$

If we make $\theta = \frac{2\pi}{n}$, as in the case of the regular n sided polygon of Geometry, we find

$$s = \frac{\rho \sin\pi \cos\left(\frac{n-1}{n}\right)\pi}{\sin\frac{\pi}{n}} = 0,$$

$$s' = \frac{\rho \sin\pi \sin\left(\frac{n-1}{n}\right)\pi}{\sin\frac{\pi}{n}} = 0.$$

Again, if we make $\theta = \frac{2m\pi}{n}$, where m is less than n, we get

$$s = \frac{\rho \sin m\pi \cos\left(\frac{n-1}{n}\right)m\pi}{\sin\frac{m\pi}{n}} = 0,$$

$$s' = \frac{\rho \sin m\pi \sin\left(\frac{n-1}{n}\right)m\pi}{\sin\frac{m\pi}{n}} = 0.$$

If we make $\theta = 2\pi$, we find

$$s = \frac{\rho \sin n\pi \cos(n-1)\pi}{\sin\pi} = \frac{0}{0},$$

$$s' = \frac{\rho \sin n\pi \sin(n-1)\pi}{\sin\pi} = \frac{0}{0}.$$

Under this form, these expressions are indeterminate: but it will appear by methods which will be investigated in a subsequent Chapter, that the value of s, under such circumstances, is $n\rho$, and that of s' is 0: it will follow therefore that the movement of the point, when $\theta = 2\pi$, is one of continual progression in the direction of its first motion.

* It appears from the first equation of figure (a) Art. 836,

$$a - a_1 \cos A_1 + a_2 \cos(A_1 + A_2) - \ldots + (-1)^{n-1} a_{n-1} \cos(A_1 + A_2 + \ldots A_{n-1}) = 0,$$

that

angles, or $(n-1)$ sides and $(n-2)$ angles, or $(n-2)$ sides and $(n-1)$ angles: in order to determine a figure, therefore, there are three classes of *data*, one class admitting of $\frac{n(n-1)(n-2)}{1.2.3}$ combinations (Art. 453), the second of $n.\frac{n(n-1)}{1.2}$ combinations and the third of the same number. When the *independent* data are fewer than $(2n-3)$ in number, the figure is not determined : if they exceed $(2n-3)$ in number, they may furnish results which are inconsistent with each other. (Arts. 402 and 403).

847. Thus, the equations of quadrilateral figures are

In the case of quadrilateral figures generally.

$$a - a_1 \cos A_1 + a_2 \cos(A_1 + A_2) - a_3 \cos(A_1 + A_2 + A_3) = 0 \quad (1),$$

$$a_1 \sin A_1 - a_2 \sin(A_1 + A_2) + a_3 \sin(A_1 + A_2 + A_3) = 0 \quad (2),$$

$$A + A_1 + A_2 + A_3 = 2\pi \quad (3).$$

If we combine equation (3) with (1) and (2) (Art. 836), we get

$$a - a_1 \cos A_1 + a_2 \cos(A_1 + A_2) - a_3 \cos A = 0 \quad (4),$$

$$a_1 \sin A_1 - a_2 \sin(A_1 + A_2) - a_3 \sin A = 0 \quad (5).$$

Of the 7 quantities a, a_1, a_2, a_3, A, A_1, A_2, if five of them be assigned within the requisite limits of value, the remaining two may be determined from the equations (4) and (5).

848. Parallelograms are defined to be quadrilateral figures *, which have their opposite sides parallel: and it will follow as a necessary consequence of this definition and from the properties of parallel lines, that the sum of every pair of consecutive angles in the figure will be equal to two right angles, and therefore that if the value of one of its angles be known or assumed, the rest are determined: for if

Properties of parallelograms.

$$A + A_1 = A_1 + A_2 = A_2 + A_3 = A_3 + A = \pi,$$

we find

$$A_1 = \pi - A, \quad A_2 = \pi - A_1 = A, \quad A_3 = \pi - A_2 = A_1,$$

that no one side can be greater than the sum of all the others, and from the relation of the exterior and interior angles, that no angle of the figure can be greater than two right angles.

* The opposite sides of regular polygons of $2m$ sides are parallel, but it is not usual to call them parallelograms, unless the figures are also quadrilateral.

and the opposite angles are therefore equal to each other: if we transfer these conditions to equations (4) and (5), we get

$$a + a_1 \cos A - a_2 - a_3 \cos A = 0,$$

$$a_1 \sin A - a_3 \sin A = 0,$$

and therefore

$$a_1 = a_3, \text{ and also } a = a_2,$$

or the opposite sides are also equal to each other.

Properties of squares.

849. Again, suppose the four sides of a quadrilateral figure to be equal to each other and one of its angles A to be a right angle, and let it be required to determine the remaining angles: under these circumstances, the equations (4) and (5) become

$$a - a \cos A_1 + a \cos (A_1 + A_2) = 0,$$

$$a \sin A_1 - a \sin (A_1 + A_2) - a = 0,$$

we thus get

$$1 = \cos A_1 - \cos (A_1 + A_2) = \quad 2 \sin \tfrac{1}{2} (2 A_1 + A_2) \sin \tfrac{1}{2} A_2,$$

$$1 = \sin A_1 - \sin (A_1 + A_2) = - 2 \cos \tfrac{1}{2} (2 A_1 + A_2) \sin \tfrac{1}{2} A_2.$$

Consequently

$$- 1 = \tan \tfrac{1}{2}(2 A_1 + A_2),$$

and therefore

$$\tfrac{1}{2} (2 A_1 + A_2) = 135^\circ. \text{ and } 2 A_1 + A_2 = 270^\circ;$$

but $$A_1 + A_2 + A_3 = 270^\circ;$$

therefore $$A_1 - A_3 = 0, \quad \text{or} \quad A_1 = A_3,$$

and therefore $$A_1 = A_2 = A_3 = 90^\circ.$$

It appears therefore that if the four sides of a quadrilateral figure be equal and one of its angles a right angle, then all its angles are right angles.

Definition of a square: superfluous.

A square is commonly defined to be a four-sided figure which has all its sides equal, and all its angles right angles: but if the proper object of the definition of a figure be the simple enunciation of the conditions which are sufficient for its determination, and no more, it is obvious that the ordinary definition of a square includes three superfluous conditions: for the same figure is determined by defining it to be that which has its four sides equal, and one of its angles a right angle: the same conclusion would follow, if we defined it to be a figure which had three sides

equal, and two of its angles right angles, or a figure which had two of its adjacent sides equal*, and three and therefore all its angles right angles: but such definitions of figures may be regarded as descriptions which comprehend their characteristic properties, which are not so much to be considered as constituting the definition of the figure as the results of independent investigations†.

850. The definitions of regular and other figures which are commonly given in Geometry, will be found similarly to involve superfluous conditions, and should rather be considered as enunciations of propositions concerning them, than as forming the essential conditions for their determination: they will be found to present themselves, in a similar manner, in the following definition of Similar Figures, which is commonly given in Geometry. Definition of similar figures.

"Similar figures are defined in Geometry to be such as have their angles severally equal to each other, and the sides about their equal angles in each proportionals."

If this definition be interpreted algebraically, it assumes, as the essential condition of the similarity of figures, that all their angles should be given, as well as the ratios of their sides: or in other words, that if a, a_1, a_2, ... a_{n-1} represent the sides, the ratios $\frac{a}{a_1}$, $\frac{a_1}{a_2}$, $\frac{a_2}{a_3}$, ... $\frac{a_{n-2}}{a_{n-1}}$ are the same in all *similar* figures, and *therefore given*, as well as the angles

$$A,\ A_1,\ A_2, \dots A_{n-1}:$$

but if the value of any one of these sides be given, such as a, all the others would be given, inasmuch as there would be $n-1$ independent equations involving them successively: it follows, therefore, that the ordinary definition of similar figures involves the fulfilment of $2n-1$ conditions, which exceeds by

* For if the four angles be right angles, the *opposite* sides are necessarily equal: the equality of two opposite sides is not, but of two adjacent sides is, an independent condition.

† One of the principal inconveniences which results from the introduction of superfluous conditions into the definitions of figures, is the necessity it imposes of shewing that the unessential as well as the essential conditions are satisfied whenever such figures present themselves in geometrical constructions and demonstrations, in order to prove that all the required conditions are satisfied: thus, in the Elements of Euclid, a figure, which has four equal sides and one angle a right angle, is not pronounced to be a square until the other angles are proved to be right angles likewise. See Euclid, Book II. Prop. 4.

2 the number of those which are requisite for the complete determination of the figure: it may be easily shewn, however, that only $2n-4$ conditions are required to determine this similarity, and that the other three are superfluous, or are not required to be included in the test by which it is ascertained.

Thus, if there be two figures of n sides, and if the $(n-2)$ ratios of the $(n-1)$ successive sides of one figure be identical with $(n-2)$ ratios of the $(n-1)$ successive sides of the other, and if the $(n-2)$ angles which they severally include be also equal to each other, then it may be shewn that all the conditions of the geometrical definition of similar figures are fulfilled

For let $a, a_1, \ldots a_{n-2}$ and x be the n successive sides of one figure, and $\alpha, \alpha_1, \ldots \alpha_{n-2}$ and x' the n successive sides of the other: and let the $(n-2)$ angles made by $a_1, a_2, \ldots a_{n-2}$ with the primitive line in one figure, and similarly by $\alpha_1, \alpha_2, \ldots \alpha_{n-2}$ in the other be $\theta_1, \theta_2, \ldots \theta_{n-2}$*: and let it be supposed that x, or the *complement* of the first figure, makes an angle ϕ, and that x', the *complement* of the second figure, makes an angle ϕ', with the primitive line: then if it be assumed that

$$\frac{a_1}{a} = \frac{\alpha_1}{\alpha}, \quad \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1}, \ldots \frac{a_{n-3}}{a_{n-2}} = \frac{\alpha_{n-3}}{\alpha_{n-2}},$$

it will follow that

$$\frac{\alpha}{a} = \frac{\alpha_1}{a_1} = \frac{\alpha_2}{a_2} \ldots = \frac{\alpha_{n-2}}{a_{n-2}} = e,$$

and therefore

$$\alpha = a e, \quad \alpha_1 = a_1 e, \quad \alpha_2 = a_2 e, \ldots \alpha_{n-2} = a_{n-2} e:$$

but, inasmuch as

$$a + a_1 (\cos\theta_1 + \sqrt{-1}\sin\theta_1) + \ldots a_{n-2}(\cos\theta_{n-2} + \sqrt{-1}\sin\theta_{n-2})$$
$$= x(\cos\phi + \sqrt{-1}\sin\phi),$$

and

$$\alpha + \alpha_1 (\cos\theta_1 + \sqrt{-1}\sin\theta_1) + \ldots \alpha_{n-2}(\cos\theta_{n-2} + \sqrt{-1}\sin\theta_{n-2})$$
$$= ae + a_1 e(\cos\theta_1 + \sqrt{-1}\sin\theta_1) + \ldots a_{n-2}e(\cos\theta_{n-2} + \sqrt{-1}\sin\theta_{n-2})$$
$$= x'(\cos\phi' + \sqrt{-1}\sin\phi') = ex(\cos\phi + \sqrt{-1}\sin\phi),$$

* For if the $n-2$ interior angles of one figure be severally equal to the $n-2$ corresponding interior angles of the other, their exterior angles are therefore the same in each, and consequently θ_1 which is the first exterior angle, and $\theta_2, \theta_3, \ldots\ldots \theta_{n-2}$, which are the sums of two, three, ... $(n-2)$ exterior angles in each, are severally the same in both figures.

we shall get

$$ex \cos \phi = x' \cos \phi',$$

$$ex \sin \phi = x' \sin \phi',$$

and therefore

$$ex = x' \text{ and } \phi = \phi'.$$

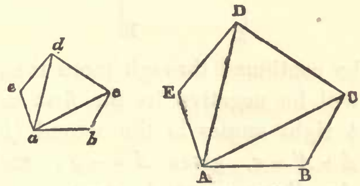
It follows, therefore, that the *complements* x and x' bear the same relation to each other which the other sides of one figure severally bear to those of the other, and also make the same angles with the primitive line: and inasmuch as $(n-1)$ angles of one figure are severally equal to $(n-1)$ angles of the other, it will likewise follow, from the equation of angles, that the remaining angle of one figure, which is contained by x and a will be equal to the remaining angle of the other figure, which is contained by x' and a: the two figures are therefore similar, in conformity with the ordinary geometrical definition.

851. In the preceding investigation, we have $(n-2)$ equations, between $(n-1)$ successive sides, and $(n-2)$ angles are assigned, making in all $(2n-4)$ conditions: and it appears that all figures are similar, which have these conditions in common: without enquiring into other consequences of the fulfilment of these conditions, we shall content ourselves with pointing out one of them, which furnishes a simple and immediate geometrical test of the similarity of rectilinear figures.

There are $2n-4$ conditions of similarity in figures of n sides.

If we suppose the ratio of AB to BC, of BC to CD, and of CD to DE to be the same as those of ab to bc, of bc to cd, and of cd to de respectively, and the angles B, C, D to be equal respectively to the angles b, c, d: then the ratio of the *complement* of AB and BC to BC, that is of AC to BC, and therefore of AC to CD is the same as that of the *complement* of ab and bc to bc, that is of ac to ab, and, therefore, of ac to cd: in the same manner, the ratio of the *complement* of AC and CD to CD, that is of AD to CD, and, therefore, of AD to DE is the same as that of the ratio of the *complement* of ac and cd to cd, that is of ad to cd, and, therefore, of ad to de: lastly, the ratio of the *complement* of AD and DE to DE, that is of AE to DE is the same as that of the *complement* of ad and de to de, that is of ae to de: it follows, therefore, that the three triangles

Geometrical test of similarity.



ABC, ACD, ADE, and *abc, acd, ade*, similarly formed in each figure, are similar to each other: and the same conclusion will equally follow, whatever be the number of the sides of the figure: it follows, therefore, that all rectilinear figures are similar to each other, in which all the triangles similarly formed in each are similar to each other.

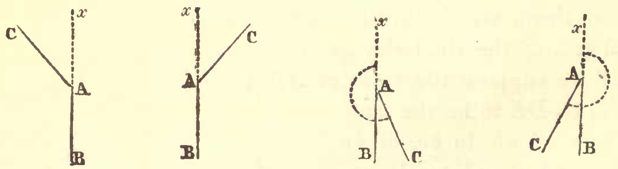
Figures with re-entrant angles may come under the general equation of figure and angles.

852. In the geometrical theory of rectilinear figures, where A represents an interior, and A' the corresponding exterior angle, we assume A and A', in the equation

$$A + A' = \pi$$

to be both of them positive, and also less than π: but figures may be conceived to be formed possessing re-entrant angles, (as we have already seen in the case of stellated figures), where one of these angles may exceed 180^0, and where this equation cannot be satisfied unless the other be negative, and conversely: thus, if the movement of transfer, in passing from the line BA to AC, be from right to left in one case (Fig. 1), and from left to right in another (Fig. 2), the external angles (xAC) generated will have different signs: and if these movements of transfer

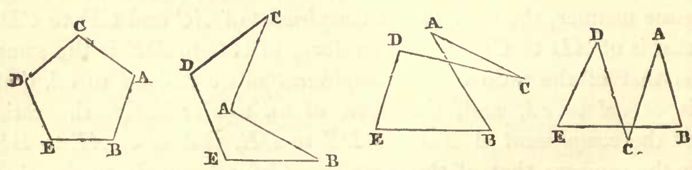
Fig. 1. Fig. 2. Fig. 3. Fig. 4.



be continued through more than 180^0, the interior angle BAC will be negative in the first case (Fig. 3), and greater than 4 right angles in the second (Fig. 4): for if $A' = \pi + \phi$, then $A + A' = \pi$, gives $A = -\phi$; and if $A' = -\pi - \phi$, then also, $A + A' \pm \pi$, gives $A = 2\pi + \phi$.

All these relations of exterior and interior angles are exem-

Fig. 1. Fig. 2. Fig. 3. Fig. 4.



plified in the formation of the equilateral pentagonal figures

(1), (2), (3) and (4), of which the first is the regular pentagon of Geometry: the second has the re-entrant and interior angle at A greater than 180^0: and therefore A' negative: the third has the angle at A negative, and the fourth the angle at A positive and greater than 4 right angles.

Such rectilinear figures, therefore, though not geometrical, satisfy the equations of figure and of angles, and they may be determined, like geometrical figures, by the aid of those equations, from the requisite data.

CHAPTER XXXII.

ON LOGARITHMS AND LOGARITHMIC TABLES AND THEIR USE.

Definition of a logarithm.

853. The index or exponent x, in the equation

$$a^x = n$$

is called the *logarithm* of n to the base a: this definition includes the primary notion of a *logarithm*, in which the term originated, which is mentioned in Art. 273.

It will be shewn in a subsequent Chapter, that if a and n are numbers {where the term number is used in its largest sense (Art. 169 and 416)} greater than 1, there is always an arithmetical value of x which satisfies the equation

$$a^x = n:$$

or, in other words, there is always an arithmetical logarithm of n to the base a.

A system of logarithms.

854. A *system* of logarithms is the series of indices of the *same base*, which correspond to the successive values of n: such a system is formed by the series of logarithms of the natural numbers from 1 to 100000, to the *base* 10, which constitute the logarithms registered in our ordinary tables, and which are therefore called *tabular logarithms*.

There are only two systems of logarithms which are commonly employed, the tabular and the Napierian.

855. Though our general reasonings have reference to systems of logarithms calculated to any base whatever, there are two systems only which are commonly employed in analysis: the first is that of tabular logarithms above referred to, and which are exclusively used in numerical calculation: the other is the system whose base is 2·7182818..., which is called Napierian, from the name of the great man to whom their invention is due, and which is almost exclusively used in analytical formulæ.

General properties of logarithms.

856. The properties of logarithms are the properties of indices of powers of the same symbol, and may be stated as follows.

Let x and x' be the logarithms of n and n' respectively to the base a, and therefore $a^x = n$ and $a^{x'} = n'$: we then find (Art. 635)

(1) $a^x \times a^{x'} = a^{x+x'} = nn'$, where $x + x'$ is the logarithm of nn': in other words, the logarithm of a product is the sum of the logarithms of its factors.

(2) $\dfrac{a^x}{a^{x'}} = a^{x-x'} = \dfrac{n}{n'}$, where $x' - x$ is the logarithm of $\dfrac{n}{n'}$: in other words, the logarithm of the quotient of two numbers is the logarithm of the dividend diminished by the logarithm of the divisor.

(3) $(a^x)^p = a^{px} = n^p$, where px is the logarithm of n^p: in other words, the logarithm of the p^{th} power of a number is p times the logarithm of the number.

(4) $(a^x)^{\frac{1}{p}} = a^{\frac{x}{p}} = n^{\frac{1}{p}}$, where $\dfrac{x}{p}$ is the logarithm of $n^{\frac{1}{p}}$: in other words, the logarithm of the p^{th} root of a number is $\dfrac{1}{p}$th part of the logarithm of the number.

857. The logarithm of the base of a system is 1. *The logarithm of the base is 1.*

For $a^1 = a$ (Art. 42), and 1 is therefore the logarithm of a, where a is the base of the system, whatever a may be.

858. The logarithm of 1, in all systems, is 0. *The logarithm of 1, in all systems, is 0.*

For $a^0 = 1$ (Art. 641), whatever be the value of a: and 0 is therefore the logarithm of 1.

859. The logarithm of 0 is infinite and negative, or $-\infty$, in all systems, where the base is greater than 1. *The logarithm of 0, when the base is greater than 1, is $-\infty$.*

For if a be greater than 1, then $a^{-\infty} = \dfrac{1}{a^{\infty}} = \dfrac{1}{\infty} = 0$: but if a be less than 1, then $a^{\infty} = 0$; and consequently, under such circumstances, ∞ is the logarithm of 0.

860. The logarithms of all numbers which are not integral powers of the base, involve a decimal part. *The logarithms of all numbers which are not integral powers of the base involve a decimal part.*

For if a be the base of the system, then the logarithms of a, a^2, a^3, a^4, or of the integral powers of a are the indices of a, and therefore whole numbers: the logarithms of all numbers

between 1 and a, a and a^2, a^2 and a^3, a^3 and a^4 have a decimal part: thus, in the tabular system, the logarithms of 10, 100, 1000, 10000, 100000, 1000000, &c., which are the articulate numbers of the decimal scale, are expressed by the series of natural numbers 1, 2, 3, 4, 5, 6, &c. being less by 1 than the number of places in each number: consequently, the tabular logarithms of all numbers

between	1	and	10	are between	0	and	1:
......	10	...	100	......	1	...	2:
......	100	...	1000	......	2	...	3:
......	1000	...	10000	......	3	...	4:
......	10000	...	100000	......	4	...	5:
......	100000	...	1000000	......	5	...	6:
......	......	......	......	......	......	......	......

and it may be observed that the integral part, or *characteristic* as it is called, of such logarithms will always be known from the number of integral places in the number, and therefore will not require to be tabulated.

The logarithms of numbers less than 1, where the base is greater than 1, are negative.

861. The logarithms of numbers less than 1, in a system whose base is greater than 1, are negative.

For, if a be greater than 1, and if x be positive, then a^x is greater than 1, and $a^{-x} = \frac{1}{a^x}$ is less than 1: the logarithms of such numbers, therefore, are necessarily negative.

Classification of the fundamental operations of Arithmetic in the order of succession and difficulty.

862. The fundamental operations of arithmetic are Addition and Subtraction, Multiplication and Division, Involution and Evolution, and the order of succession in which they are thus arranged is likewise the order of the difficulty and labour of performing them, the inverse operations of Subtraction, Division, and Evolution being generally less simple, direct, and expeditious than the corresponding operations of Addition, Multiplication, and Involution.

The order of the corresponding logarithmic operations is lower by two unities.

863. If we compare the operations with numbers with the corresponding operations with their logarithms, it will be observed that their *order*, with reference to the preceding classification, is reduced by two unities: thus the operation of Multiplication with numbers corresponds to that of Addition with logarithms,

Division to that of Subtraction, Involution to that of Multiplication, and Evolution to that of Division: and it follows as a consequence of this reduction of the order of arithmetical operations when logarithms are employed, that the operations of Addition and Subtraction with numbers have no corresponding operations with logarithms, and therefore cannot be effected by means of them, without the aid of expedients which will be afterwards described.

It is this reduction of the order of arithmetical operations which gives to logarithms their great value and importance, and which enables us to bring within the range of the resources of calculation, arithmetical processes which would otherwise become practically impossible, from their complexity and from the labour which they would require.

864. In the following examples, which are introduced for the purpose of illustrating the more simple applications of the general properties of logarithms to reduce the order of arithmetical operations, it is assumed that the logarithms of all numbers, to the base 10, whether *whole* or *decimal,* are given in the tables. Examples.

(1) To find the product of 371 and 796 by logarithms.

$$\begin{array}{lrl} \text{Log} & 371 = & 2.5693739 \\ \text{log} & 796 & \underline{2.9009131} \\ \text{log} & 295316 = & 5.4702870 \end{array}$$

We find, from the tables, the logarithms of 371 and 796, and add them together: we then find the number 295316, of which this sum is the logarithm, which is therefore the product of 371 and 796.

(2) To find the quotient of 400107 divided by 197.

$$\begin{array}{lrl} \text{Log} & 400107 = & 5.6021762 \\ \text{log} & 197 & \underline{2.2944662} \\ \text{log} & 2031 = & 3.3077100 \end{array}$$

We find the logarithms of the dividend 400107, and of the divisor 197, and subtract the second from the first: the remainder, from this subtraction, is the logarithm of 2031, which is therefore the quotient required.

(3) To find the value of 9^9, or of the ninth power of nine. The logarithm of 9^9 is 9 times the logarithm of 9.

$$\begin{array}{rl} \text{Log} \quad 9 = & .9542425 \\ & \quad\quad\quad 9 \\ \hline \log 387420489 = & 8.5881825 \end{array}$$

$$\text{or} \quad 387420489 = 9^9.$$

It is proper to observe that the ordinary tables will only give the first seven digits of this number.

(4) To find the square root of 67943.

$$\text{Log} \quad 67943 = 4.8321447$$

$$\log \quad \sqrt{67943} = \tfrac{1}{2} \log 67943 = 2.4160723$$

$$\log \ 260.6587 = 2.4160723$$

Consequently

$$\sqrt{67943} = 260.6587\ldots$$

(5) To find the cube root of 67943.

$$\text{Log} \quad 67943 = 4.8321447$$

$$\log \quad \sqrt[3]{67943} = \tfrac{1}{3} \log 67943 = 1.6107149$$

$$\log \ 40.80515 = 1.6107149$$

Consequently

$$\sqrt[3]{67943} = 40.84515\ldots$$

(6) To find the 7th root of 67943.

$$\text{Log} \quad 67943 = 4.8321447$$

$$\log \quad \sqrt[3]{67943} = \tfrac{1}{7} \log 67943 = \ .6903064$$

$$\log \ 4.901245 = .6903064$$

Consequently

$$\sqrt[7]{67943} = 4.901245\ldots$$

This is an operation, which is extremely laborious and difficult, when effected without the aid of logarithms.

Superior convenience and brevity of tables of logarithms to base 10.

865. If the logarithms which are registered in tables were coextensive with the numbers which are employed in arithmetical operations, we should experience no difficulty in reducing the order of such operations by two unities, whatever was the base

of the system which was adopted: but a very little consideration would be sufficient to shew that the extent of tables which would be required for a succession of all numbers, both whole and decimal, even if confined within the narrowest limits which are required for the purposes of calculation, would be much too great to be easily registered or referred to: but it will readily appear from the relation which exists between the characteristics of tabular logarithms and the articulate numbers of the decimal scale, which we have noticed above (Art. 860), that the logarithms, in that system, of all numbers, whether whole or decimal, expressed by the formulæ $N \times 10^n$ and $\frac{N}{10^n}$, or in other words, of all numbers *which are expressed by the same succession of significant digits*, may be found from *one* opening of the tables*.

Thus

$$\log 96498 = 4.9845228,$$

$$\log 96498 \times 10 = \log 964980 = 5.9845228,$$

$$\log 96498 \times 100 = \log 9649800 = 6.9845228,$$

$$\log \frac{96498}{10} = \log 9649.8 = 3.9845228,$$

$$\log \frac{96498}{100} = \log 964.98 = 2.9845228,$$

$$\log \frac{96498}{1000} = \log 96.498 = 1.9845228,$$

$$\log \frac{96498}{10000} = \log 9.6498 = .9845228,$$

$$\log \frac{96498}{100000} = \log .96498 = \bar{1}.9845228,$$

$$\log \frac{96498}{1000000} = \log .096498 = \bar{2}.9845228,$$

$$\log \frac{96498}{10000000} = \log .0096498 = \bar{3}.9845228.$$

* For $\log N \times 10^n = \log N + \log 10^n = n + \log N,$

$$\log \frac{N}{10^n} = \log N - \log 10^n = \log N - n:$$

for the logarithm of 10^n, if 10 be the base, is n.

In the three last cases the sign – is placed above, and not before, the characteristic, which is alone affected by it, the decimal part of the logarithm, or *mantissa,** remaining positive: but if we subtract the *mantissa* from 1, forming what is called its *arithmetical complement,* and diminish the negative characteristic by 1, we shall obtain the correct negative logarithm corresponding: we thus find

$$\log\ .96498 = -\ .0154772,$$

$$\log\ .096498 = -1.0154772,$$

$$\log\ .0096498 = -2.0154772.$$

Tables of logarithms give *mantissæ* only.

866. The tables, therefore, of a system of logarithms, whose base is coincident with the base of our scale of arithmetical notation, give the *mantissæ* only of logarithms in one column, with the significant digits, in their proper order, of the corresponding numbers in another, inasmuch as the *characteristic* may be always supplied from the number of integral places, or in case there are none, from the position of the decimal point with respect to the first significant digit: thus, the *mantissa* of the logarithm of 53399 is .7275331, which alone is recorded in the tables, the complete logarithm 4.7275331 being at once supplied by prefixing a *characteristic* which is less by 1 than the number of integral places in the proposed number: and in a similar manner, we find, from the same *mantissa,* the complete logarithm of .00053399, which is $\bar{4}.7275331$, where the negative characteristic $\bar{4}$ exceeds by 1 the number of *zeros,* which immediately follow the decimal point.

Inconveniences of tables of logarithms to a base which is different from 10.

867. If, however, the base of the system was not coincident with the radix or base of our system of arithmetical notation, the logarithms of all numbers included in the formulæ $10^n \times N$ and $\frac{N}{10^n}$ and which are therefore expressed by the same succession of significant digits, would not be known from one opening of the tables: thus if the base was the number 2.7182828 (Art.

* This term was introduced by Euler, and may be conveniently used to designate the decimal part of a logarithm, in the absence of any simple designation which can be supplied by our own language.

855), and if we assume the prefix log to designate Napierian and not tabular logarithms *, we should find

$$\begin{aligned}
\log 964 &= 6.8710911,\\
\log 9640 &= \log 964 + \log 10\\
&= 6.8710911 + 2.3025851\\
&= 9.1736762,\\
\log 96400 &= \log 964 + \log 100\\
&= 6.8710911 + 4.6051702\\
&= 11.4762613,\\
\log 96.4 &= \log 964 - \log 10\\
&= 6.8710911 - 2.3025851\\
&= 4.5685060.
\end{aligned}$$

It thus appears that we cannot pass from the Napierian logarithm of N to that of $N \times 10^n$ or $\frac{N}{10^n}$ without an additional opening of the tables for the purpose of finding the Napierian logarithm of 10^n.

Specimen of a table of logarithms.

868. The following is a specimen of the form of our ordinary tables of the logarithms of numbers.

N	0	1	2	3	4	5	6	7	8	9	D	Prop.
2650	423 2459	2623	2786	2950	3114	3278	3442	3606	3770	3933		163
51	4097	4261	4425	4589	4753	4916	5080	5244	5408	5571		1 16 2 33 3 49 4 65 5 82 6 98 7 114 8 130 9 147
52	5735	5899	6063	6226	6390	6554	6718	6881	7045	7209	163	
53	7372	7536	7700	7864	8027	8191	8355	8518	8682	8846		
54	9009	9173	9336	9500	9664	9827	9991	$\overline{0}154$	0318	0482		
55	424 0645	0809	0972	1136	1300	1463	1627	1790	1954	2117		

* Some modern writers are accustomed to subscribe the base to the prefix log, in order to designate the particular system of logarithms which is employed: thus $\log_{10}$ means tabular logarithms: $\log_e$ means Napierian logarithms, whose base is $e = 2.7182818$, and similarly in other cases: such refinements of notation, however, are rarely necessary, as the circumstances under which they occur will generally indicate the nature of the logarithms, whether tabular or Napierian, which are used.

In the first column are placed, underneath the letter *N*, the first four digits of the number, the fifth being written in the same line with it at the head of the successive columns. In the column headed 0, are written the *mantissæ* of the logarithms of 26500, 26510, 26520,......, the three first digits being suppressed as long as they remain the same with those in the first line: in the column headed 1, are written the four last digits of the *mantissæ* of the logarithms of 26511, 26521, 26531, ...: in the column headed 2, the four last digits of the *mantissæ* of the logarithms of 26512, 26522, 26532, ...: and so on for the remaining columns headed by the remaining nine digits, in their order.

It will be observed that the *mantissa* of the logarithm of 26546 is 4239991 and that of the logarithm of 26547 is 4240154: this change of the *third* digit of the *mantissa* from 3 to 4, is indicated, in the place where it first occurs, by writing the four last digits thus $\bar{0}154$.

The logarithms of large numbers increase very nearly in the same proportion with the numbers themselves.

869. If the *mantissæ* of the logarithms of the successive numbers, which are given in the Extract from the tables in the last Article, be subtracted from each other, their difference will appear to be between .0000164 and .0000163, and will be found, by actual reference of the Tables, to continue nearly the same for the whole series of numbers between 26500 and 26800: we are authorized to conclude, therefore, that the *mantissæ* of large numbers will increase, *within small limits, very nearly* in the same proportion with the numbers themselves, and that, consequently, if the *mantissæ* of the logarithms of two successive large numbers, and therefore *their difference*, be given, the *mantissæ* of the logarithms of all intermediate numbers may be found approximately, if not accurately, by a simple proportion and conversely*: this property of logarithms is very important, as furnishing us with very simple and expeditious methods of greatly extending the range of the tables.

To find the logarithm of a number of more than five places.

870. Thus, if it was required to find the logarithm of a number of more than 5 digits (and therefore beyond the range of the ordinary tables), we should proceed as follows.

We should find from the tables, the logarithm of the number

* This proposition will be easily shewn to be an immediate consequence of the logarithmic series, which will be given in Chapter XXXV.

formed by the first five digits; and to this we should add, from the *column of proportional parts* headed *Prop.*, the number corresponding to the 6th digit and $\frac{1}{10}$th of the number corresponding to the 7th digit, by advancing the number given in the column one place to the right: their sum would be the logarithm required to seven places of figures: and if the number of decimal places of the logarithms recorded in the Tables was greater than what is commonly given, the same method might be extended to find the logarithms of numbers of more than seven places*.

Thus, if it was required to find the logarithm of 265.4678, we find from the tables Example.

Log 265.46	= 2.4239991
Prop. part of logarithm corresponding to digit 7 in the 6th place	114
Similarly for digit 8 in the 7th place	13
The logarithm of 265.4678	= 2.4240118

871. Again, if it was required to find the number corresponding to a non-tabulated logarithm, we should proceed as follows: To find the number corresponding to a logarithm not in the tables.

We find the difference between the *mantissa* of the given logarithm and the next inferior *mantissa* in the tables and we write down the corresponding number: in the column of proportional parts, we find the number which is equal to, or next below, the difference thus found and the digit opposite to it is the 6th digit of the number sought for: if there be a remainder, we subjoin a zero to it, forming a new difference (removed one place to the right) and the digit in the column of proportional parts which is opposite to the number equal to or next below, it, is the 7th digit of the number required; and similarly, if more digits than 7 are required to be determined.

* For if D be the difference in the *mantissæ* corresponding to a unit of the tabular number, $\frac{D}{10}$ and $\frac{D}{100}$ will be the differences, in conformity with the property assumed in the text, corresponding to the two next inferior units in the 6th and 7th places respectively, and therefore $\frac{aD}{10}$ and $\frac{bD}{100}$ will be these differences, if a and b be the digits in those places: the first is found in the column of proportional parts opposite the digit a: the second, removed one place to the right, is found in the same column opposite the digit b.

Example.

Thus, let it be required to find the number, whose logarithm is 1.233678, and whose precise value is not found in the Tables.

The given logarithm is	1. 233678
The next less tabular-logarithm of 17.122 is	1.2335545
Their difference is	133.
The table of prop. parts gives the digit 5 for	127
Their difference (6) followed by *zero* gives	60
The same table gives 2 for 51	51
Their difference (9) followed by *zero* gives	90
The same table gives 3 for 76	76
Their difference 14 followed by *zero* gives	140
The same table gives 5 for 127	127

The number required is therefore 17.1225235 nearly.

Tables of logarithmic sines, cosines, tangents, &c.

872. Inasmuch as the sines, cosines, tangents, secants, &c. of angles enter into formulæ which are the subjects of calculation equally with other symbols possessing assigned numerical values, a register of the logarithms of their successive values becomes equally necessary with that of the logarithms of the series of natural numbers. Tables or *canons* of *natural* sines, (Chap. XXVIII.) cosines, tangents, cotangents, secants, cosecants, will contain their successive values for every minute (and in some tables for every ten seconds) of all angles between $1'$ and 45^0 and consequently between $1'$ and 90^0, if taken in an inverse order, when the sine is replaced by the cosine, the tangent by the cotangent, and the secant by the cosecant: whilst tables of the logarithms of sines, cosines, tangents and cotangents, secants and cosecants, will contain the logarithms of their *natural* values *increased by the number* 10, arranged in precisely the same order, each page of the *natural* values being opposite to a page of their corresponding *logarithmic* values.

The reason why the logarithms of the numerical values of the sines, cosines, &c. are increased by 10.

873. A very little consideration will shew the great convenience, for the purposes of calculation, of increasing the logarithms of the goniometrical quantities, as recorded in the tables, by the number 10: for the natural values of the sines and cosines are included between 0 and 1 and the *characteristics* of their logarithms are therefore necessarily negative: thus

$\sin 1' = .0002909$: its unaltered logarithm is $\bar{4}.4637261$,

$\sin 1^\circ = .0174524$; .... $\bar{2}.2418553$,

$\sin 50^\circ = .7660444$; .... $\bar{1}.8842540$,

$\cos 30' = .9999619$; .... $\bar{1}.9999836$,

$\cos 30^\circ = .8660254$; .... $\bar{1}.9375306$,

$\cos 85^\circ = .0871557$; .... $\bar{2}.9402960$.

If such logarithms, therefore, were registered in tables, their *haracteristics* and *mantissæ* would have different signs and great confusion might thus be occasioned in the calculation, by means of them, of the values of formulæ in which such quantities occurred, particularly in the hands of calculators, as is very commonly the case, who have no very accurate knowledge of the principles of Algebra: it is for this reason, that the logarithms of all goniometrical quantities are increased by 10, and the logarithms of $\sin 1'$, $\sin 1^\circ$, $\sin 50^\circ$, $\cos 30'$, $\cos 30^\circ$, $\cos 85^\circ$, present themselves, therefore, in the tables, with the following values:

$$6.4637261,\ 8.2418553,\ 9.8842540,$$
$$9.9999836,\ 9.9375306,\ 8.9402960.$$

874. The number 10 is the logarithm of 10^{10} and the *registered* or *tabulated* logarithms of the sines, cosines, and other goniometrical quantities are the *proper* logarithms of the products of those quantities and 10^{10}; or if we adopt the ancient definitions, we should say that the tabulated logarithms were those of the sines, cosines, &c. in a circle whose radius was 10^{10} or 10000000000 (Art. 754, Note): but in the adaptation of formulæ to logarithmic computation, we may always pass from the *tabulated* to the *proper* logarithms of such quantities, by subtracting 10 from their characteristics and conversely: or we may follow the practice, which is generally most expeditious and convenient, of adding to or subtracting from, the final logarithm which results, 10 or any multiple of 10, which may obviously have been introduced by the use of *tabulated* instead of *proper* logarithms.

Transition from the tabulated logarithms of sines, cosines, &c. to their natural logarithms and conversely.

The student will find, in the two following chapters, several examples not merely of the adaptation of expressions, involving goniometrical quantities to logarithmic computation, but likewise of the calculation of their numerical values by means of them.

CHAPTER XXXIII.

ON THE DETERMINATION OF THE SIDES AND ANGLES OF TRIANGLES OR TRIGONOMETRY PROPERLY SO CALLED.

Equations which express the general relations of the sides and angles of triangles.

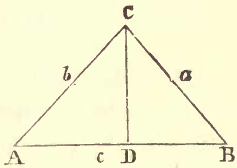
875. We have already shewn (Art. 833), that if a, b, c be the sides of a triangle, and A, B, C the angles opposite to them, the following equations will express their general relations to each other, assuming c to be the primitive line.

$$c - a \cos B - b \cos A = 0 \qquad (1),$$

$$a \sin B - b \sin A = 0 \qquad (2),$$

$$A + B + C = \pi.^* \qquad (3).$$

* These equations may be derived immediately from the triangle, by the aid of the definitions of the sine and cosine: for if ABC be a triangle and if CD be drawn perpendicular to AB, then we have



$$AD = AC \cos A = b \cos A,$$

$$\text{and} \quad BD = BC \cos B = a \cos B;$$

and therefore

$$c = AB = AD + DB = b \cos A + a \cos B, \text{ or}$$

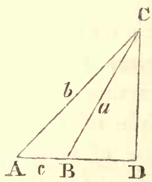
$$c - a \cos B - b \cos A = 0, \qquad (1)$$

which is the first equation. And again, since $CD = b \sin A = a \sin B$, we get

$$a \sin B - b \sin A = 0, \qquad (2),$$

which is the second equation.

If the triangle be oblique-angled, as in Fig. 2, and if CD be drawn perpendicular upon AB produced, then the angle $CBD = \pi - B$, and we find



$$AD = AC \cos A = b \cos A, \text{ and}$$

$$BD = BC \cos CBD = a \cos (\pi - B) = - a \cos B,$$

(Art. 763) and therefore

$$AB = c = AD - BD = b \cos A + a \cos B,$$

$$\text{or} \quad c - a \cos B - b \cos A = 0; \qquad (1)$$

and again, since

$$CD = AC \sin A = a \sin A$$

$$= CB \sin CBD = a \sin (\pi - B) = a \sin B, \qquad \text{(Art. 763)}.$$

we find

$$a \sin B - b \sin A = 0. \qquad (2).$$

Of the six elements which present themselves in every triangle, namely, the three sides and the three angles, there are five involved in the first of the preceding equations, four in the second, and three in the third. The last equation, which is the equation of angles, determines the third angle when two of them are given: it follows, therefore, that two only of the three angles can be considered as dependant upon the specific* conditions of the triangle.

Two only of the three angles of a triangle furnish specific conditions for its determination.

876. The solution of the second of the preceding equations, which involves two sides and the sines of their opposite angles, enables us at once to express one of these four elements in terms of the other three.

Of the four elements, the two sides and their opposite angles, any three will determine the fourth.

We thus find

$$a = \frac{b \sin A}{\sin B}, \quad b = \frac{a \sin B}{\sin A},$$

$$\sin A = \frac{a}{b} \sin B, \quad \sin B = \frac{b}{a} \sin A.$$

Likewise, if in equation (1),

$$c - a \cos B - b \cos A = 0,$$

we replace b by $\dfrac{a \sin B}{\sin A}$, we get

$$c = a \cos B + \frac{a \sin B}{\sin A} \cos A$$

$$= \frac{a(\sin A \cos B + \sin B \cos A)}{\sin A}$$

$$= \frac{a \sin(A+B)}{\sin A} = \frac{a \sin C}{\sin A};$$

for, by equation (3), we find

$$\sin(A+B) = \sin(\pi - C) = \sin C.$$

877. The conclusions in the last Article may be exhibited under the form

The sides of a triangle are proportional to the sines of the opposite angles.

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}, \quad \frac{b}{c} = \frac{\sin B}{\sin C}:$$

* The specific conditions are those which are not common to all triangles, but peculiar to that which is under consideration: the three sides and two angles, when given, constitute specific conditions, provided the sum of no two sides exceeds the third and the two angles together do not exceed 180°: if three angles be given, they are not the angles of a triangle, unless their sum be 180°: but if two of them only, whose sum is less than 180°, be given, there is always a triangle of which they may form two of the angles.

and it follows, therefore, that the *sides of a triangle are proportional to the sines of the opposite angles.*

Expression for one side of a triangle in terms of the two other sides and the cosine of their included angle.

878. The fundamental equations in Art. 875, will readily furnish us with an expression for one side of a triangle in terms of the two other sides and of the cosine of the angle which they include, which is of fundamental importance in the science of Trigonometry, properly so called.

For from equation (1), we get

$$c^2 = (a \cos B + b \cos A)^2 = a^2 \cos^2 B + b^2 \cos^2 A + 2ab \cos A \cos B.$$

From equation (2), we similarly get

$$0 = (a \sin B - b \sin A)^2 = a^2 \sin^2 B + b^2 \sin^2 A - 2ab \sin A \sin B.$$

Consequently, by adding, we find

$$\begin{aligned} c^2 &= a^2 (\cos^2 B + \sin^2 B) + b^2 (\cos^2 A + \sin^2 A) \\ &\qquad + 2ab (\cos A \cos B - \sin A \sin B) \\ &= a^2 + b^2 + 2ab \cos (A + B), \\ &= a^2 + b^2 - 2ab \cos C: \qquad (a), \end{aligned}$$

$$\text{for } \cos^2 B + \sin^2 B = 1, \ \cos^2 A + \sin^2 A = 1,$$

$$\cos A \cos B - \sin A \sin B = \cos (A + B) = \cos (\pi - C) = - \cos C.$$

In a similar manner, we shall find

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad (b),$$

$$b^2 = a^2 + c^2 - 2ac \cos B \qquad (c),$$

Expression for the cosine of an angle of a triangle in terms of its sides.

879. If we severally solve the equations (*a*), (*b*), and (*c*) with respect to $\cos C$, $\cos A$, and $\cos B$, we shall find

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad (d),$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad (e),$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad (f),$$

which are expressions for the cosines of the several angles of a triangle in terms of its sides.

Expression for the sine of an angle it terms of its sides.

880. Corresponding and more symmetrical expressions may be found for the sines of the angles of a triangle in terms of its

sides, and which present themselves likewise in a form which is better suited, as will afterwards appear, to logarithmic computation: we proceed to obtain them, as follows:

From equation (a), Art. 878, we get

$$2ab\cos C = a^2 + b^2 - c^2.$$

If we add $2ab$ to both sides, we find

$$\begin{aligned} 2ab(1+\cos C) &= a^2 + 2ab + b^2 - c^2 \\ &= (a+b)^2 - c^2 \\ &= (a+b+c)(a+b-c). \quad \text{Art. 66.} \quad (g). \end{aligned}$$

Again, if from $2ab$ we subtract $2ab\cos C$ on one side, and $a^2 + b^2 - c^2$ on the other, we get

$$\begin{aligned} 2ab(1-\cos C) &= c^2 - a^2 + 2ab - b^2 \\ &= c^2 - (a^2 - 2ab + b^2) \\ &= c^2 - (a-b)^2 \\ &= (a+c-b)(b+c-a), \qquad (h). \end{aligned}$$

If we multiply the two sides of the equations (g) and (h) respectively together, we find

$$4a^2b^2(1-\cos^2 C) = (a+b+c)(b+c-a)(a+c-b)(a+b-c).$$

Replacing $1-\cos^2 C$ by $\sin^2 C$, dividing both sides by $4a^2b^2$, and extracting the square roots, we get

$$\sin C = \frac{\sqrt{\{(a+b+c)(b+c-a)(a+c-b)(a+b-c)\}}}{2ab}.$$

If we make $s = \dfrac{a+b+c}{2}$, we find $a+b+c = 2s$.

$$\begin{aligned} b+c-a &= b+c+a-2a = 2s-2a = 2(s-a), \\ a+c-b &= a+c+b-2b = 2s-2b = 2(s-b), \\ a+b-c &= a+b+c-2c = 2s-2c = 2(s-c), \end{aligned}$$

and therefore

$$\sin C = \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{ab} \qquad (i).$$

In a similar manner, we find

$$\sin A = \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{bc} \qquad (k),$$

$$\sin B = \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{ac} \qquad (l).$$

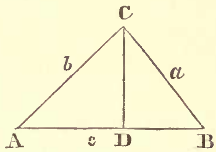
The expressions for sin A, sin B, and sin C, have the same numerator, which is twice the area of the triangle.

881. It will be observed, that the several expressions for sin A, sin B, and sin C, have the same numerator

$$2\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

which likewise expresses twice the area of the triangle.

For the area of the triangle ABC is equal to one-half of the rectangle contained by its base AB and the perpendicular CD let fall upon it from the opposite angle C, and it is therefore expressed (Art. 592) by $\frac{1}{2}AB \times CD = \frac{1}{2}c \times CD = \frac{bc \sin A}{2} = \frac{ac \sin B}{2}$, since $CD = a \sin B = b \sin A$.



If in $\frac{bc \sin A}{2}$, we replace $\sin A$ by the expression given in the last Article, we get the area of the triangle $(k) = \sqrt{\{s(s-a)(s-b)(s-c)\}}$, or it is equal to the square root of the continued product of the semi-sum of the sides and of its several excesses above the three sides of the triangle.

The preceding expressions are adequate for the solution of all the cases of Trigonometry.

882. If it be granted that the value of an angle may be both determined and calculated from the value of its sine, cosine, tangent, &c. (and the limits of the values of the angles of triangles will be found to remove all ambiguity when the value of the angle is not to be determined from the value of the sine*), the expressions given in the preceding Articles will enable us to calculate the sides, angles, and areas of triangles whenever the *data* are adequate to determine them.

We shall now proceed to consider the different cases of *data* which are sufficient to determine a triangle, and to exemplify their logarithmic computation.

Given one side and two angles, to find the remaining parts.

883. Case 1. Given a, A and B, to find C, b and c.

The third angle C is found from the equation

$$A + B + C = \pi,$$

which gives $C = \pi - (A + B)$.

* For the sine of an angle is identical in magnitude and sign with the sine of its supplement; but the cosine, tangent, and cotangent of an angle are severally equal to the cosine, tangent and cotangent of its supplement in magnitude, but different from it in sign.

The equations in Art. 876, give us

$$b = \frac{a \sin B}{\sin A} \text{ and } c = \frac{a \sin C}{\sin A}.$$

The same equations, adapted to logarithmic computation, become

$$\log b = \log a + \log \sin B - \log \sin A,$$
$$\log c = \log a + \log \sin C - \log \sin A.$$

Let $a = 749.6$, $A = 37^{\circ}.14'$, and $B = 67^{\circ}.27'$: to find b, c and C. Example.

In the first place, to find C, we have

$$\begin{array}{rl} \pi = & 180^{\circ} \\ \hline A = & 37.14' \\ B = & 67.27' \\ \hline A + B & 104.41' \\ \hline C = & 75^{\circ}.19' \end{array}$$

To find b, we have

$$\begin{array}{rl} \log 749.6 = & 2.8748296 \\ \log \sin 67^{\circ}.27' = & 9.9654582 \\ \hline & 12.8402878 \\ \log \sin 37^{\circ}.14' = & 9.7818002 \\ \hline \log 1144.16 = & 3.0584876 \text{ or } b = 1144.16. \end{array}$$

The use of the arithmetical complement of $\log \sin 37^{\circ}.14'$, which is found by subtracting it from 10, will enable us somewhat to simplify this process,

$$\begin{array}{rl} \log 749.6 = & 2.8748296 \\ \log \sin 67^{\circ}.27' = & 9.9654582 \\ \text{arith. comp. } \log \sin 37^{\circ}.14' = & .2181998 \\ \hline \log 1144.16 = & 3.0584876 \end{array}$$

We reject 10 from the final characteristic, as forming simply the *tabulated augment* of the logarithm of $\sin 67^{\circ}.27'$.

To find c, we have

$$\begin{array}{rl} \log 749.6 = & 2.8748296 \\ \log \sin 75^{\circ}.19' = & 9.9855798 \\ \text{arith. comp. } \log \sin 67^{\circ}.27' = & .2181998 \\ \hline \log 1197.42 = & 3.0786092 \text{ or } c = 1197.42. \end{array}$$

Given two sides and an angle opposite to one of them.

884. Case 2. Given a, b and A: to find B, C and c. The equations in Art. 876 give us

$$\sin B = \frac{b \sin A}{a},$$

which becomes, when adapted to logarithmic computation,

$$\log \sin B = \log \sin A + \log b - \log a.$$

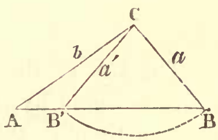
When A and B are known, we find C and c as in the first case.

The solution is ambiguous when the given angle is opposite to the less of the two sides.

If the value of B is to be determined from that of $\sin B$, it is uncertain whether it is B or $\pi - B$, or whether it is greater or less than 90^0, unless the connexion of the data with other properties of the triangle remove the ambiguity.

Thus if b be less than a, or if the angle whose value is sought for is opposite to the less of the two given sides, then B is less than A and therefore necessarily less than 90^0: in this case, therefore, there is no ambiguity *.

But if b be greater than a, or if the angle, whose value is sought for, is opposite to the greater of the two given sides, then there are two triangles, possessing precisely the same data, in which the angle opposite to the greater side in one of them is supplemental, and therefore equisinal to the corresponding angle in the other; and in this case the solution, therefore, is necessarily ambiguous.



Thus if A, which is the given angle, be opposite to a, the less of the two given sides a and b, then if we describe an arc of a circle with centre C and radius CB, it will cut AB produced in two points, B and B', forming two triangles CAB and CAB', whose angles B and B', opposite to the common side AC or b, are supplemental, and therefore equisinal to each other: but the two triangles formed have the same data, which are CB, CA, and the angle A in one triangle, and $CB_1 = CB$, CA and the angle A in the other.

* For the greater side of a triangle is opposite to the greater angle and conversely, a conclusion which is proved in Geometry (Euclid, Book I. Prop. 11.) and may be easily deduced from the fundamental equations in Art. 875: for if a be greater than b, then $\sin A$ is greater than $\sin B$: if A be greater than 90^0, then B is less than 90^0 and therefore less than A, inasmuch as the sum of the angles A and B is less than 180^0: but if A be less than 90^0, then B must be less than 90^0, and also less than A: for if B be greater than 90^0, its supplement $\pi - B$ must be less than 90^0, and therefore less than A, since $\sin(\pi - B)$ is less than $\sin A$: but if $\pi - B$ be less than A, then π is less than $A + B$, which is impossible: it follows therefore, that B is necessarily less than A, or that the greater side is opposite to the greater angle.

The ambiguity, therefore, which we have shewn to exist in the case under consideration is not a consequence of the imperfection of our formula, but is essentially dependent upon the conditions of the problem proposed.

The sides of a triangle are 17.09 and 93.451 and the angle opposite to the greater of them is $93^\circ.16'$: to find the angle opposite to the less. Example.

The logarithmic formula is

$$\log \sin B = \log \sin A + \log b - \log a.$$

$$\begin{array}{lr}
\log \sin 93^\circ.16 = \log \sin 86^\circ.44' = & 9.9992938 \\
\log 17.09 & = 1.2327421 \\
\hline
 & 11.2320359 \\
\log 93.451 & 1.9705840 \\
\hline
\log \sin 10^\circ.31' & 9.2614519^{*}
\end{array}$$

If the conditions of the problem had been reversed, and if the sides 17.09, 93.451, and the angle $10^\circ.31'$ opposite to 17.09 had been given, to find the angle opposite to the greater side 93.451, the operation would have stood as follows:

$$\begin{array}{lr}
\log \sin 10^\circ.31' = & 9.2614519 \\
\log 93.451 & 1.9705840 \\
\hline
 & 11.2320359 \\
\log 17.09 & 1.2327420 \\
\hline
\log \sin 86^\circ.44' \text{ or } \log \sin 93^\circ.16' = & 9.9992939
\end{array}$$

and both these angles will equally answer the conditions of the problem.

885. Case 3. Given a, b and C: to find c, A and B. Given two sides and the included angle.

The equation (a) in Art. 878, gives us

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

* Since -1.9705840 is equivalent to $\bar{2}.0294160$, (Art. 865) we may put the example in the text under the following form

$$\begin{array}{r}
\log \sin 93^\circ.16' = 9.9992938 \\
\log 17.09 = 1.2327421 \\
\text{arith. comp. } \log 93.451 = 8.0294160 \\
\hline
\log \sin 10^\circ.31' = 9.2614519
\end{array}$$

rejecting 10 from the characteristic.

where c is expressed in terms of the given quantities a, b and $\cos C$.

When c is found, A, and therefore B, is determined as in Case 2.

Meaning of the phrase "adapted to logarithmic computation."

The expression $a^2 + b^2 - 2ab \cos c$, whose terms are connected by the signs + and −, and whose values cannot be ascertained without the performance of arithmetical operations of considerable difficulty and labour, is said to be *not adapted to logarithmic computation*: and formulæ generally are said to be *adapted or not adapted to logarithmic computation*, when they consist of products, quotients, powers or roots of easily calculated terms, or which do not require a mixed application of logarithmic and numerical computation in order to determine their values.

Its vagueness.

The phrase, however, and its usage, is somewhat vague and indefinite, inasmuch as it does not determine absolutely the *conditions of greatest convenience* in the selection or preparation of formulæ for the purposes of computation: in other words, a formula which is *not adapted to logarithmic computation*, according to the technical usage of the term, may admit, in many cases, of more rapid computation, by mixed or even by merely arithmetical means, than one which is so: the selection therefore of one or the other, when more than one method are within our reach, must be determined by the judgment and experience, and sometimes by the *taste* of the computer.

Solution of the proposed problem without the use of subsidiary angles.

The introduction of subsidiary angles will enable us to adapt the expression

$$a^2 + b^2 - 2ab \cos C$$

or any other, whose terms are connected by the signs + and −, to logarithmic computation, and we shall reserve for the following Chapter, which will be devoted to their theory and use, the further consideration of the various methods which are employed for this purpose; in the mean time the following method will enable us to solve the problem proposed.

Since $\frac{a}{b} = \frac{\sin A}{\sin B}$ (Art. 877), we have (Art. 792, No. 8)

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)},$$

but $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$ is known, since C is given: it follows therefore that

$$\tan\left(\frac{A-B}{2}\right) = \frac{(a-b)}{(a+b)} \cot \frac{C}{2}$$

is known, and therefore $\frac{A-B}{2}$ is known: consequently

$$A = \frac{A+B}{2} + \frac{A-B}{2} \text{ and } B = \frac{A+B}{2} - \frac{A-B}{2}$$

are known, when c may be found by the methods given in Art. 876.

Let the two sides of the triangle be 27.04 and 74.67, and let the angle included between them be $117^{\circ}.20'$; and let it be required to find the remaining side and angles. Example.

In this case

$$a = 74.67,$$

$$b = 27.04,$$

and therefore

$$a - b = 47.63 \text{ and } a + b = 101.71,$$

$$A + B = \pi - C = 180^{\circ} - 117^{\circ}.20' = 62^{\circ}.40',$$

and therefore

$$\frac{A+B}{2} = 31^{\circ}.20',$$

$$\log\tan\left(\frac{A-B}{2}\right) = \log\tan\left(\frac{A+B}{2}\right) + \log(a-b) - \log(a+b),$$

$\log\tan 31^{\circ}.20'$	=	9.7844784
$\log 47.63$	=	1.6778806
		11.4623590
$\log 101.71$	=	2.0073637
$\log\tan 15^{\circ}.55'$	=	9.4549953

$$\frac{A+B}{2} = 31^{\circ}.20',$$

$$\frac{A-B}{2} = 15^{\circ}.55'.$$

Therefore $A = 47^\circ.15'$, and $B = 15^\circ.25'$.

Also $c = \dfrac{a \sin C}{\sin A}$ and therefore

$$\log c = \log 74.67 + \log \sin 117^\circ.20' - \log \sin 47^\circ.15'$$

$$\begin{array}{lr}
\log 74.67 & = \ 1.8731462 \\
\log \sin 117^\circ.20' = \log \sin 62^\circ.40' = & 9.9485852 \\
\hline
 & 11.8217314 \\
\log \sin 47^\circ.15' & = \ 9.8658868 \\
\hline
\log 90.333 & = \ 1.9558446
\end{array}$$

We have thus determined the remaining side and angles of the triangle.

Given three sides.

886. Case 4. Given a, b and c to find A, B and C: and also the area of the triangle.

The cosines (Art. 879) of the three angles of the triangle are expressed by

$$\cos A = \frac{b^2 + c^2 - a}{2bc},$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

The expressions for the cosines of the angles, though not ambiguous, are not adapted to logarithmic computation.

which are not in a form adapted to logarithmic computation, inasmuch as the separate terms of which the numerators are composed cannot be ascertained without a logarithmic, or a somewhat laborious arithmetical, computation.

It should be observed however, that the three angles are determined by these formulæ without ambiguity: for the cosine of the angle will be positive or negative according as it is less or greater than 90°.

If we make $k = \sqrt{\{s(s-a)(s-b)(s-c)\}}$, where $s = \dfrac{a+b+c}{2}$,

and where the factors are formed therefore by a very easy and rapid arithmetical process, we shall find (Art. 880)

The expressions for the sines of the angles, though ambiguous, are adapted to logarithmic computation.

$$\sin A = \frac{2k}{bc}, \quad \sin B = \frac{2k}{ac} \text{ and } \sin C = \frac{2k}{ab},$$

which are in a form adapted to logarithmic computation.

We thus get

$$\log \sin A = \log k + 10 + \log 2 - (\log b + \log c)$$

$$= \tfrac{1}{2}\{\log s + \log (s-a) + \log (s-b) + \log (s-c)\}$$

$$+ 10 + \log 2 - (\log b + \log c).$$

The log sin A, which is here sought for, is the tabulated logarithm, exceeding by 10 the proper logarithm of $\frac{2k}{bc}$.

We have already shewn (Art. 881), that k is the area of the triangle, expressed in units which are the squares described upon the linear units in terms of which the sides are expressed, whether inches, feet, yards or miles.

It should be kept in mind that the angle A determined by this formula is ambiguous; but inasmuch as only one angle of the triangle can be greater than 90^0, and the greatest angle is opposite to the greatest side, this ambiguity is confined to that angle alone; and this angle will be greater or less than 90^0, according as the expression for its cosine $\frac{b^2+c^2-a^2}{2bc}$ is negative or positive: or in other words, according as b^2+c^2 is less or greater than a^2.

Again, if the three angles A, B and C be determined from the preceding formulæ, and if, assuming them to be acute, their sum or $A+B+C=\pi$, then their values are correctly determined: but if not, this equation will be satisfied, (assuming that the arithmetical and logarithmic processes are correctly performed), by taking that value of the equisinal (Art. 776) angle opposite the greatest side, which is greater than 90^0.

Example.

Let the three sides of the triangle be 107·9, 193·4 and 217·12 yards: and let it be required to find the three angles of the triangle and its area.

$$
\begin{array}{rl}
a & = 107.9 \\
b & = 193.4 \\
c & = 217.12 \\
\hline
2) & 518.42 \\
\hline
s & = 259.21 : \log s = 2.4136518 \\
a & = 107.9 \\
\hline
s - a & = 151.31 : \log (s-a) = 2.1798676 \\
s & = 259.21 \\
b & = 193.4 \\
\hline
s - b & = 65.81 : \log (s-b) = 1.81882919 \\
\hline
s & = 259.21 \\
c & = 217.12 \\
\hline
s - c & = 42.09 : \log (s-c) = 1.6241789
\end{array}
$$

$$
\begin{array}{lr}
 & 2)\ 8.0359902 \\
\log k = \log 10423.06 & = 4.0179951 \\
10 + \log 2 & = 10.3010300 \\
\hline
 & 14.3190251 \\
\hline
\log b = \log 193.4 & = 2.2844565 \\
\log c = \log 217.12 & = 2.3366998 \\
\hline
\log b + \log c & = 4.6211563 \\
\hline
\log \sin 29^{\circ}.46' = \log \sin A & = 9.6958686
\end{array}
$$

Again,

$$
\begin{array}{lr}
10 + \log 2 + \log k & = 14.3190251 \\
\hline
\log a = \log 107.9 & = 2.0330214 \\
\log c = \log 217.12 & = 2.3366998 \\
\hline
\log a + \log c & = 4.3697212 \\
\hline
\log \sin 62^{\circ}.51' = \log \sin B & = 9.9493039
\end{array}
$$

The angles A and B being found, the angle C is known: or it may be found as follows:

$$10 + \log 2 + \log k = 14.3190251$$

$$\begin{aligned} \log a = \log 107.9 \quad &= 2.0330214 \\ \log b = \log 193.4 \quad &= 2.2864565 \\ & 4.3194779 \\ \log \sin 87^\circ.23' = \log \sin C &= 9.9995472 \end{aligned}$$

Therefore

$$\begin{aligned} A &= 29^\circ.46' \\ B &= 62^\circ.51' \\ C &= 87^\circ.23' \\ & \overline{180^\circ} \end{aligned}$$

Also $k = 10423.06$ square yards $= 2.1535$ acres, which is the area of the triangle.

Modifications of the preceding formulæ might be proposed, which, in some cases, would enable us to obtain the required results, by shorter and more expeditious processes than those which are given above*: but it is not our object, in this Chapter, to give a complete treatise of practical Trigonometry, but merely to explain generally the method by which we adapt our formulæ, in one of their most useful applications, to arithmetical and logarithmic computation.

* Thus, it will readily follow, from the investigation in Art. 880, that

$$1 + \cos C = 2\cos^2\frac{C}{2} = \frac{2s(s-c)}{ab},$$

$$1 - \cos C = 2\sin^2\frac{C}{2} = \frac{2(s-b)(s-c)}{ab},$$

and therefore

$$\tan\frac{C}{2} = \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.$$

If therefore we make

$$R = \sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s}\right\}},$$

a symmetrical expression with respect to a, b, c, (and equal to the radius of the circle inscribed in the triangle) we get

$$\tan\frac{C}{2} = \frac{R}{s-c}, \quad \tan\frac{B}{2} = \frac{R}{s-b}, \quad \tan\frac{A}{2} = \frac{R}{s-a},$$

expressions which are not only more easily calculated than the expressions in the text, but are also free from ambiguity. See "Geometrical Problems and Analytical Formulæ, with their application to Geodetical Problems;" p. 21, a very ingenious and original work by the late Professor Wallace, of Edinburgh.

CHAPTER XXXIV.

ON THE USE AND APPLICATION OF SUBSIDIARY ANGLES.

What is meant by a subsidiary angle.

887. A *subsidiary* angle is one whose sine, cosine, tangent, &c. does not exist in the primitive formula, but which is introduced for the purpose of modifying its form, or of facilitating its computation, by means of logarithms or otherwise.

Used in the adaptation of formulæ to logarithmic computation which are not otherwise adaptible.

A *subsidiary* angle is generally necessary in the computation of expressions consisting of two or more terms connected with the signs + and −, and which cannot otherwise be adapted to logarithmic computation, as will more fully appear in many of the examples which follow.

In the case of $a + b$.

888. Let the expression to be adapted to logarithmic computation be $a+b$, where a and b are positive quantities, whose separate numerical values are not easily found or added together, without the aid of logarithms.

In the first place

$$a + b = a\left(1 + \frac{b}{a}\right):$$

if we make $\tan^2\theta = \frac{b}{a}$ *, where θ is the *subsidiary* angle, we shall find

$$a + b = a\,(1 + \tan^2\theta) = a\sec^2\theta = \frac{a}{\cos^2\theta}:$$

and therefore

$$\log(a+b) = \log a + 20 - 2\log\cos\theta$$

$$= \log a + 2 \text{ arith. com. } \log\cos\theta.$$

Example.

Thus, as an example, let it be required to find the value of the expression

* Therefore $\log\tan\theta = 10 + \frac{1}{2}(\log b - \log a)$.

$$397 \cos 14^\circ + 410.7 \sin 67^\circ.10'$$

log 397	= 2.5987905	
log cos 14°	= 9.9869041	
	12.5856946	
log b (rejecting 10)		= 2.5856946
log 410.7	= 2.6135247	
log sin 67°. 10′	= 9.9645602	
	= 12.5780849	
log a (rejecting 10)		= 2.5780849
log b − log a =		2) .0076097
log tan 45°. 15′		= 10.0038048
arith. com. log cos 45°. 15′		= .1524183
		2
2 arith. com. log cos 45°. 15′		= .3048366
log a		= 2.5780849
log 770.56 = log $(a+b)$		= 2.8829215

889. Let the expression be $a-b$, where both a and b are positive, and a greater than b. In the case of $a-b$.

In the first place, we find

$$a-b=a\left(1-\frac{b}{a}\right),$$

where $\frac{b}{a}$ is less than 1: we may assume, therefore,

$$\sin^2\theta=\frac{b}{a},$$

which gives

$$\log\sin\theta=10-\tfrac{1}{2}(\log a-\log b):$$

we thus get

$$a-b=a(1-\sin^2\theta)=a\cos^2\theta,$$

and therefore

$$\log(a-b)=\log a+2\log\cos\theta-20.$$

In a similar manner, if we make $\frac{b}{a} = \cos^2\theta$, we get

$$a - b = a(1 - \cos^2\theta) = a\sin^2\theta,$$

and therefore

$$\log(a - b) = \log a + 2\log\sin\theta - 20^*.$$

In the case of $\frac{a-b}{a+b}$.

890. Let the expression be of the form $\frac{a-b}{a+b}$, where both a and b are positive and a greater than b.

If we make $\frac{b}{a} = \tan\theta$, we get

$$\frac{a-b}{a+b} = \frac{1 - \frac{b}{a}}{1 + \frac{b}{a}} = \frac{1 - \tan\theta}{1 + \tan\theta} = \tan(45^\circ - \theta).$$

Thus the expression (Art. 886),

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\tan\left(\frac{A+B}{2}\right)$$

* Thus, if two sides a and b and the included angle C of a triangle be given, we find

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin\{\pi - (B + C)\}}{\sin B} = \frac{\sin(B + C)}{\sin B},$$

and therefore, dividing by $\sin C$,

$$\frac{a}{b\sin C} = \frac{\sin(B + C)}{\sin B\sin C} = \cot B + \cot C,$$

and consequently

$$\cot B = \frac{a}{b\sin C} - \cot C$$
$$= \frac{a}{b\sin C}\left(1 - \frac{b\cos C}{a}\right).$$

If we make $\frac{b\cos C}{a} = \cos^2\theta$, we get

$$\cot B = \frac{a}{b\sin C}(1 - \cos^2\theta) = \frac{a\sin^2\theta}{b\sin C}.$$

In a similar manner, we shall find, making $\cos^2\theta' = \frac{a\cos C}{b}$,

$$\cot A = \frac{b\sin^2\theta'}{a\sin C}.$$

This is a very convenient and expeditious method of solving a triangle when two sides and the included angle are given : see Prof. Wallace's Geometrical Theorems and Analytical Formulæ, &c. Edinburgh, 1839. p. 42.

becomes

$$\tan\left(\frac{A-B}{2}\right) = \tan(45^\circ - \theta)\tan\left(\frac{A+B}{2}\right),$$

if we make $\tan\theta = \dfrac{b}{a}$.

891. Let the expression be $\sqrt{(a^2-b^2)}$, where a is greater than b. In the case of $\sqrt{(a^2-b^2)}$.

Make $\dfrac{b}{a} = \cos\theta$, and we get

$$\sqrt{(a^2-b^2)} = a\sqrt{(1-\cos^2\theta)} = a\sin\theta.$$

Thus the expression in Art. 885 or

$$\begin{aligned} c &= \sqrt{(a^2+b^2-2ab\cos C)} \\ &= \sqrt{\{a^2+2ab+b^2-2ab(1+\cos C)\}} \\ &= (a+b)\sqrt{\left\{1-\frac{2ab(1+\cos C)}{(a+b)^2}\right\}} \\ &= (a+b)\sqrt{\left\{1-\frac{4ab\cos^2\frac{C}{2}}{(a+b)^2}\right\}} \\ &= (a+b)\sqrt{(1-\cos^2\theta)} = (a+b)\sin\theta, \end{aligned}$$

making $\cos\theta = \dfrac{2\sqrt{ab}\,.\cos\frac{C}{2}}{a+b}$.

In this case

$$\log\cos\theta = \log\cos\frac{C}{2} + \log 2 + \tfrac{1}{2}(\log a + \log b) - \log(a+b)$$

$$\log c = \log(a+b) + \log\sin\theta - 10.$$

892. Let the expression be $\sqrt{(a^2+b^2)}$. In the case of $\sqrt{(a^2+b^2)}$.

If we make $\tan\dfrac{\theta}{2} = \dfrac{b}{a}$, we get

$$\sqrt{(a^2+b^2)} = a\sqrt{(1+\tan^2\theta)} = a\sec\theta$$

$$= \frac{a}{\cos\theta}.$$

Thus the expression in Art. 885, or

$$c = \sqrt{(a^2 + b^2 - 2ab\cos C)}$$
$$= \sqrt{(a^2 - 2ab + b^2 + 2ab - 2ab\cos C)}$$
$$= (a-b)\sqrt{\left\{1 + \frac{2ab(1-\cos C)}{(a-b)^2}\right\}}$$
$$= (a-b)\sqrt{\left\{1 + \frac{4ab\sin^2\frac{C}{2}}{(a-b)^2}\right\}}$$
$$= (a-b)\sqrt{(1+\tan^2\theta)}$$
$$= \frac{(a-b)}{\cos\theta},$$

making $\tan\theta = \dfrac{2\sqrt{ab}\,.\sin\frac{C}{2}}{a-b}$.*

In the case of $a \pm \sqrt{(a^2 \pm b^2)}$.

893. Let the expressions be of the form

$$a \pm \sqrt{(a^2 - b^2)},$$
$$\text{or}\quad a \pm \sqrt{(a^2 + b^2)}.$$

If, in the first of these expressions, we make

$$\frac{b}{a} = \sin\theta,$$

we get

$$a + \sqrt{(a^2 - b^2)} = a(1+\cos\theta) = 2a\cos^2\frac{\theta}{2},$$

$$a - \sqrt{(a^2 - b^2)} = a(1-\cos\theta) = 2a\sin^2\frac{\theta}{2}\,†.$$

* The same expression is also adapted to logarithmic computation by making

$$\tan\theta = \frac{a+b}{a-b}\tan\frac{C}{2},$$

which gives

$$c = (a-b)\cos\frac{C}{2}\sec\theta = \frac{\left(a - b\cos\frac{C}{2}\right.}{\cos\theta}.$$

† The roots of the equation

$$x^2 - px + q = 0$$

are $$\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} - q\right)}:$$

If, in the second, we make

$$\frac{b}{a} = \tan\theta,$$

we get

$$a + \sqrt{(a^2+b^2)} = b(\cot\theta + \operatorname{cosec}\theta) \text{ (Art. 791)}$$

$$= b\cot\frac{\theta}{2},$$

$$a - \sqrt{(a^2+b^2)} = b(\cot\theta - \operatorname{cosec}\theta) \text{ (Art. 791)}$$

$$= -b\tan\frac{\theta}{2}.*$$

894. Let the expression consist of a series of terms connected with the sign +, such as In the case of a series of terms connected with the sign +.

$$a + b + c + d + \&c.$$

In the first place,

$$a + b = a\sec^2\theta, \text{ if } \tan^2\theta = \frac{b}{a}.$$

Again, replacing $a\sec^2\theta$ by a', we find

$$a + b + c = a' + c = a'\sec^2\theta' = a\sec^2\theta\sec^2\theta',$$

$$\text{if } \tan^2\theta' = \frac{c}{a'} = \frac{c}{a+b} = \frac{c}{a\sec^2\theta}.$$

Similarly, replacing $a\sec^2\theta\sec^2\theta'$ by a'', we find

$$a + b + c + d = a'' + d = a''\sec^2\theta''$$

$$= a\sec^2\theta\sec^2\theta'\sec^2\theta'',$$

and if we make

$$\frac{2\sqrt{q}}{p} = \sin\theta,$$

the two roots will be expressed by

$$p\cos^2\frac{\theta}{2} \text{ and } p\sin^2\frac{\theta}{2}.$$

It is assumed that q is less than $\frac{p^2}{4}$.

* The roots of the equation

$$x^2 - px - q = 0$$

$$\text{are } \frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)};$$

$$\text{and if we make } \frac{2\sqrt{q}}{p} = \tan\theta,$$

the two roots will be expressed by

$$\sqrt{q}\cot\frac{\theta}{2} \text{ and } -\sqrt{q}\tan\frac{\theta}{2}.$$

$$\text{if}\quad \tan^2\theta'' = \frac{d}{a''} = \frac{d}{a+b+c} = \frac{d}{a\sec^2\theta\sec^2\theta'},$$

and so on, whatever be the number of terms of the series.

When the series is convergent and the terms connected with the signs + and −.

895. Let the expression consist of series of *converging* terms connected alternately with the signs + and −, such as

$$a - b + c - d + \dots$$

In the first place

$$a - b = a\cos^2\theta = a', \ \text{if}\ \frac{b}{a} = \sin^2\theta.$$

Again

$$a - b + c = a' + c = \frac{a'}{\cos^2\theta'} = \frac{a\cos^2\theta}{\cos^2\theta'},$$

$$\text{if}\ \frac{c}{a'} = \frac{c}{a-b} = \tan^2\theta'.$$

If we include a fourth term, we get

$$a - b + c - d = a'' - d = a''\cos^2\theta'' = \frac{a\cos^2\theta\cos^2\theta''}{\cos^2\theta'} = a''',$$

$$\text{if}\ \frac{d}{a''} = \frac{d}{a-b+c} = \sin^2\theta''.$$

For five terms, we get

$$a - b + c - d + e = a''' + e = \frac{a'''}{\cos^2\theta'''} = \frac{a\cos^2\theta\cos^2\theta''}{\cos^2\theta'\cos^2\theta'''},$$

and similarly for any number of terms.

In the case of a continued product of factors of the form $a+b$ or $a-b$.

896. Let the expression be a continued product, such as

$$(a+b)(a'+b')(a''+b'')\dots$$

$$\text{or}\ (a-b)(a'-b')(a''-b'').$$

In the first case we make

$$\tan^2\theta = \frac{b}{a},\quad \tan^2\theta' = \frac{b'}{a'},\quad \tan^2\theta'' = \frac{b''}{a''},\dots$$

and we get

$$(a+b)(a'+b')(a''+b'')\dots = \frac{a\,a'\,a''\dots}{\cos^2\theta\cos^2\theta'\cos^2\theta''\dots}.$$

In the second, we make

$$\sin^2\theta = \frac{b}{a}, \quad \sin^2\theta' = \frac{b'}{a'}, \quad \sin^2\theta'' = \frac{b''}{a''}, \dots$$

and we get

$$(a-b)(a'-b')(a''-b'') = a\,a'\,a'' \cos^2\theta \cos^2\theta' \cos^2\theta''.$$

897. There are many applications of analysis, particularly in Astronomy, in which it will be found to be extremely useful to convert expressions, such as Conversions of other formulæ by means of subsidiary angles.

$$a \sin A \pm b \cos A,$$

$$a \cos A \pm b \sin A,$$

into equivalent expressions of the form

$$\alpha \sin (A \pm \theta), \quad \text{or} \quad \alpha \cos (A \pm \theta),$$

by means of the subsidiary angle θ.

Thus, if we make $\alpha = \sqrt{(a^2 + b^2)}$, $\frac{a}{\alpha} = \cos\theta$ and $\frac{b}{\alpha} = \sin\theta$, we get

$$a \sin A \pm b \cos A = \alpha \sin\theta \sin A \pm \alpha \sin\theta \cos A$$

$$= \alpha \sin (A \pm \theta) = \sqrt{(a^2 + b^2)} \sin (A \pm \theta),$$

and

$$a \cos A \pm b \sin A = \alpha \cos\theta \cos A \pm \alpha \sin\theta \sin A$$

$$= \alpha \cos (A \mp \theta) = \sqrt{(a^2 + b^2)} \cos (A \mp \theta).$$

An expression of the form

$$a \sin^2 A - b,$$

where b is less than $a \sin^2 A$, becomes, by making $b = a \sin^2\theta$,

$$a \sin^2 A - a \sin^2\theta = a \sin (A + \theta) \sin (A - \theta),$$

a modification of its form which is not unfrequently used.

CHAPTER XXXV.

ON EXPONENTIAL AND LOGARITHMIC SERIES.

Developement of a^x. 898. THE exponential expression a^x, the series into which it may be developed, and the various logarithmic and other series which may be deduced from it, enter very extensively into analytical enquiries, and deserve the most careful examination of the student. We shall begin with the developement of a^x into an equivalent series.

If, in a^x, we replace a by $1+(a-1)$, making

$$a^x = \{1+(a-1)\}^x,$$

we shall find, by the binomial theorem, (Art. 680)

$$a^x = 1 + x(a-1) + x(x-1)\frac{(a-1)^2}{1.2} + x(x-1)(x-2)\frac{(a-1)^3}{1.2.3} + \&c.:$$

if we actually multiply the factors of the *exponential* coefficients (Art. 688), and collect together the terms which severally involve the same powers of x, denoting their successive coefficients by k, A_2, A_3, A_4, &c., we shall get

$$a^x = 1 + kx + A_2\frac{x^2}{1.2} + A_3\frac{x^3}{1.2.3} + \dots \qquad (1),$$

$$\text{where } k = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \dots{}^*$$

a series which possesses, as we shall afterwards shew (Art. 906), some important properties; it remains to determine the coefficients A_2, A_3, ...

* For the last terms of these several products are

$$-x, \quad 1\times 2x, \quad -1\times 2\times 3x, \quad 1\times 2\times 3\times 4x,$$

which, when severally divided by the terminal (Art. 688) coefficients 1×2, $1\times 2\times 3$, &c. become x, $\frac{-x}{2}$, $\frac{x}{3}$, $\frac{-x}{4}$, ...

If in equation (1), we replace x by y, we get

$$a^y = 1 + ky + A_2 \frac{y^2}{1 \cdot 2} + A_3 \frac{y^3}{1 \cdot 2 \cdot 3} + \ldots \qquad (2).$$

If we multiply together the two sides of the equations (1) and (2), we get

$$\begin{aligned} a^x \times a^y = a^{x+y} = 1 + kx + A_2 \frac{x^2}{1 \cdot 2} &+ A_3 \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots \\ + ky + k^2 xy &+ k A_2 \frac{x^2 y}{1 \cdot 2} + \ldots \qquad (3), \\ + A_2 \frac{y^2}{1 \cdot 2} &+ k A_2 \frac{xy^2}{1 \cdot 2} + \ldots \\ &+ A_3 \frac{y^3}{1 \cdot 2 \cdot 3} + \ldots \end{aligned}$$

But if, in equation (1), we replace x by $x + y$, we find

$$a^{x+y} = 1 + k(x+y) + A_2 \frac{(x+y)^2}{1 \cdot 2} + A_3 \frac{(x+y)^3}{1 \cdot 2 \cdot 3} + \ldots$$

which becomes, by expanding the successive powers of $x + y$ which it involves,

$$\begin{aligned} a^{x+y} = 1 + kx + A_2 \frac{x^2}{1 \cdot 2} &+ A_3 \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots \\ + ky + 2 A_2 \frac{xy}{1 \cdot 2} &+ 3 A_3 \frac{x^2 y}{1 \cdot 2 \cdot 3} + \ldots \qquad (4), \\ + A_2 \frac{y^2}{1 \cdot 2} &+ 3 A_3 \frac{xy^2}{1 \cdot 2 \cdot 3} + \ldots \\ &+ A_3 \frac{y^3}{1 \cdot 2 \cdot 3} + \ldots \end{aligned}$$

We thus obtain two forms of the developement of a^{x+y}, which can only become identical in *form* as well as in *value*, when the respective terms of both series which involve x and y *similarly* are identical with each: thus if we compare together the terms in the two series which involve y only, without its powers, we find

$$k^2 xy = \frac{2 A_2 xy}{1 \cdot 2} \quad \text{or } A_2 = k^2,$$

$$\frac{k A_2 x^2 y}{1 \cdot 2} = \frac{3 A_3 x^2 y}{1 \cdot 2 \cdot 3} \quad \text{or } A_3 = k A_2 = k^3,$$

$$\frac{k A_3 x^3 y}{1 \cdot 2 \cdot 3} = \frac{4 A_4 x^3 y}{1 \cdot 2 \cdot 3 \cdot 4} \quad \text{or } A_4 = k A_3 = k^4,$$

if we replace A_2, A_3, A_4, ... by their values, we get

$$a^x = 1 + kx + \frac{k^2 x^2}{1 \cdot 2} + \frac{k^3 x^3}{1 \cdot 2 \cdot 3} + \dots$$

a series in which k and x are similarly involved.

Conditions of identity of equivalent series.

899. The principle which is involved in this investigation is one of great importance, and capable of very extensive application: it is presumed that the form of the series which is equivalent to a^x is independent of the specific value of x, and therefore the same when x is replaced by y, or by $x+y$, or by any symbol or combination of symbols whatsoever: but if a^{x+y} be equivalent to the product of a^x and a^y for all values of x and y, then likewise the series for a^{x+y} must be equivalent to the product of the series for a^x and a^y under the same circumstances: and this equivalence of the results which are obtained implies that they are identical in all those terms in which x and y, one or both, are similarly involved: we are thus enabled to obtain a series of equations, expressing the conditions of identity, by which the form, or analytical values of the successive coefficients are determined.

Is the existence of the series for a^x necessary or not?

900. It may be further observed that the existence of an equivalent series for a^x, or of a series which shall possess the same analytical properties with a^x, is a necessary consequence of the binomial series in its general form, and involves "the principle of the permanence of equivalent forms" no further than it is involved in the binomial theorem: but when the binomial or any other theorem is once established, whatever be the principle upon which it rests, it becomes one of the known and acknowledged results of Symbolical Algebra, and may be employed in the deduction or establishment of other conclusions equally with the results of the definitions of Arithmetical Algebra: it is thus that the bases of Symbolical Algebra are perpetually enlarged, and the great principles which present themselves in the first and most elementary of the generalizations which it requires, are speedily replaced by other and successive links in the long chain of consequences which are found in the progress of our enquiries: such results, therefore, though their existence *per se* may not be necessary, yet become necessary results when considered with reference to each other and to the propositions upon which they are finally dependent.

901. Resuming the consideration of the equation The numerical value of $a^{\frac{1}{k}}$.

$$a^x = 1 + kx + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} + \dots \qquad (a),$$

we shall find, by making $x = \frac{1}{k}$,

$$a^{\frac{1}{k}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \dots$$

a rapidly converging series, from which its numerical value may be calculated to any required degree of accuracy: the aggregation of 14 of its terms gives us

$$a^{\frac{1}{k}} = 2.7182818,$$

which is correct as far as the last figure: but no finite decimal number can express its accurate value*.

It is usual, in all cases, to denote this number 2.7182818 by the symbol e; it is the base of Napierian logarithms (Art. 855), which are exclusively used in analytical formulæ.

902. If, in equation (a), we replace x by $\frac{x}{k}$, we get The series for e^x.

$$a^{\frac{x}{k}} = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \dots :$$

* The process for this purpose is very simple and expeditious: divide 1, and the successive quotients which thence arise by the successive natural numbers: the sum of the quotients, increased by 2, is the number required: thus,

	1.
2)	1.
3)	.5
	.16666666
4)	.04166666
5)	.00833333
6)	.00138888
7)	.00019841
8)	.00002480
9)	.00000276
10)	.00000027
11)	.00000003
12)	.00000000
	2.71828180

It has been shewn, in Art. 203, that e is an *incommensurable* number: it is not capable, therefore, of being expressed by any finite decimal.

but $a^{\frac{1}{k}} = e$, and therefore $a^{\frac{x}{k}} = e^x$: we thus get

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots \qquad (b).$$

This series, which constantly presents itself in analysis, becomes convergent from its r^{th} term, if r is the first whole number greater than x.

The Napierian logarithm of a.

903. Since $a^{\frac{1}{k}} = e$, we get

$$a = e^k,$$

and consequently k is the logarithm of a to base e, or, in other words, it is the Napierian logarithm of a.

Modulus of a system of logarithms.

904. Again, since $a = e^k$, we find

$$a^x = e^{kx} = n,$$

where kx is the Napierian logarithm of n, whilst x is its logarithm to the base a: if we multiply, therefore, Napierian logarithms by $\frac{1}{k}$ or $\frac{1}{\log a}$, we shall obtain the corresponding logarithms to base a; and the multiplier $\frac{1}{k}$ is called the *modulus* (μ) of that system.

The series for $\log n$ in terms of n.

905. Again, since $k = \log a$, we get (Art. 890)

$$\log a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \ldots$$

or replacing a by n,

$$\log n = (n-1) - \frac{(n-1)^2}{2} + \frac{(n-1)^3}{3} - \frac{(n-1)^4}{4} + \&c. \qquad (c),$$

a series of great importance in analysis, and which expresses the Napierian logarithm of a number in terms of the number itself.

The series for $\log n$ may be made convergent in all cases.

906. This series is divergent when n exceeds 2; but a very simple modification of its form will enable us to make it, in all cases, as rapidly convergent as we please.

For $(n^{\frac{1}{m}})^m = n$ and $\log (n^{\frac{1}{m}})^m = m \log (n^{\frac{1}{m}}) = \log n$, and therefore

$$\log n = m \log (n^{\frac{1}{m}}) = m\{(n^{\frac{1}{m}} - 1) - \tfrac{1}{2}(n^{\frac{1}{m}} - 1)^2 + \tfrac{1}{3}(n^{\frac{1}{m}} - 1)^3 - \ldots\} \quad (d).$$

If we suppose n to be greater than 1, it is always possible, by assuming m sufficiently large, to make $n^{\frac{1}{m}}$ exceed 1 by a quantity as small as we choose: thus, if $n = 10$, we find

$$10^{\frac{1}{2}} - 1 = 2.162277,$$

$$10^{\frac{1}{4}} - 1 = .778279,$$

$$10^{\frac{1}{8}} - 1 = .333521,$$

$$10^{\frac{1}{16}} - 1 = .154781,$$

$$10^{\frac{1}{32}} - 1 = .074607\ldots$$

$$10^{\frac{1}{2^{32}}} - 1 = .000000000536112,$$

which, multiplied into 2^{32}*, gives

$$2^{32}\left(10^{\frac{1}{2^{32}}} - 1\right) = 2.3025851,$$

a result which expresses the accurate value of the Napierian logarithm of 10, as far as it goes: for it will be found that the second term of the series (d), which is $2^{31}\left(10^{\frac{1}{2^{32}}} - 1\right)^2$, possesses no significant digit in the first 8 places of decimals.

907. The reciprocal of log 10 or The modulus of tabular logarithms.

$$\frac{1}{\log 10} = \frac{1}{2.3025851} = .434294481,$$

is the *modulus* (Art. 904) of tabular logarithms, or it is the factor by which Napierian logarithms, when calculated by the series in the last Article or by other and more expeditious methods, require to be multiplied, in order to reduce them to the corresponding logarithms of the tables.

908. The preceding method of calculating tables of logarithms is theoretically perfect, but the operations of multiplication, division and extraction of roots which it involves, makes Practical objections to the preceding method of calculating logarithms.

* Briggs, in his *Arithmetica Logarithmica*, has found

$$10^{\frac{1}{2^{54}}} - 1 = .00000000000000012781914932003 5:$$

the same work contains the tabular logarithms of the natural numbers as far as 1097 to 61 places of decimals, an unrivalled monument of labour and ingenuity, at a period when the various expedients which modern analysis supplies for abridging such calculations, were almost entirely unknown.

its application for such purposes extremely tedious and embarrassing: but in the calculation of the logarithms of successive numbers, we shall be enabled to resort to much more expeditious methods, founded upon logarithmic series, or upon the *differences* of successive logarithms, some of which we shall proceed to notice.

Logarithmic series.

909. Since (Art. 905)

$$\log n = (n-1) - \tfrac{1}{2}(n-1)^2 + \tfrac{1}{3}(n-1)^3 - \dots \qquad (c),$$

we shall find, by replacing n by $1+x$,

Log $(1+x)$.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad (e),$$

a series of very simple form, and which is very frequently referred to.

910. If, in this equation (e), we replace x by $-x$, we get

Log $(1-x)$.

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \qquad (f).$$

911. If we subtract the series for $\log(1-x)$ from that for $\log(1+x)$, we shall get

Log $\frac{1+x}{1-x}$.

$$\log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) \qquad (g).$$

Log $\frac{p}{q}$.

912. If in this equation (g), we make $\frac{1+x}{1-x} = \frac{p}{q}$, and therefore $x = \frac{p-q}{p+q}$, we get

$$\log\frac{p}{q} = 2\left\{\frac{p-q}{p+q} + \tfrac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \tfrac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \&c.\right\} \qquad (h).$$

When $p = q+1$.

If we make $p = q+1$, we get $\frac{p-q}{p+q} = \frac{1}{2q+1}$, and therefore

$$\log\frac{q+1}{q} = 2\left\{\frac{1}{2q+1} + \tfrac{1}{3}\,\frac{1}{(2q+1)^3} + \&c.\right\},$$

or

$$\log(q+1) = \log q + 2\left\{\frac{1}{2q+1} + \tfrac{1}{3}\,\frac{1}{(2q+1)^3} + \&c.\right\} \qquad (i).$$

This is a series adapted to the calculation of the logarithms of successive numbers, being rapidly convergent in all cases, and more particularly so, when the numbers, whose logarithms are sought for, are large.

913. If p and q are resolvible into factors which involve the first power of a number n *only* without a coefficient, and if the fraction $\frac{p-q}{p+q}$ be of such a form as to decrease rapidly when n increases, then we may express the logarithm of the greatest of the factors of p and q, in terms of the logarithms of the other factors, and of the series (h), Expression of the logarithm of a number in terms of the logarithms of inferior numbers, and of the series for $\log \frac{p}{q}$.

$$2\left\{\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \&c.\right\}.$$

Thus, let $p = x^2$ and $q = x^2 - 1 = (x+1)(x-1)$, and therefore $\frac{p-q}{p+q} = \frac{1}{2x^2-1}$: we find $\log \frac{p}{q} = 2\log x - \log(x+1) - \log(x-1)$, and therefore

$$\log(x+1) = 2\log x - \log(x-1)$$
$$-2\left\{\frac{1}{2x^2-1} + \frac{1}{3}\frac{1}{(2x^2-1)^3} + \dots\right\},$$

by which the logarithm of a number is expressed in terms of the logarithms of the two next inferior numbers, and of a rapidly converging series.

Again, if we assume

$$p = x^2(x-7)^2(x+7)^2 = x^6 - 98x^4 + 2401x^2,$$
$$q = (x-3)(x+3)(x-5)(x+5)(x-8)(x+8)$$
$$= x^6 - 98x^4 + 2401x^2 - 14400,$$

we find $\frac{p-q}{p+q} = \frac{7200}{x^6 - 98x^4 + 2401x^2 - 7200}$,

and therefore

$$\log(x+8) = 2\log(x+7) + 2\log x + 2\log(x-7)$$
$$-\log(x+5) - \log(x+3) - \log(x-3) - \log(x-5)$$
$$-\log(x-8) - 2\left\{\frac{7200}{x^6 - 98x^4 + 2401x^2 - 7200} + \&c.\right\}.$$

If $x = 100$, the first term of the series, or

$$\frac{7200}{x^6 - 98x^4 + 2401x^2 - 7200}$$

has no significant digit in the first 7 places of decimals: for this and higher values of x therefore it may be altogether neglected, and the calculation of the logarithms of a considerable succession of such numbers may be made dependent upon the logarithms of 8 inferior numbers only.

Differences of the logarithms of successive numbers.

914. The practical and most expeditious method of calculating the logarithms of successive numbers will be found to be dependent upon the Theory of Differences, which will form the subject of a subsequent Chapter of this work: we shall notice it no further in this place, than is requisite to justify the rules which are given in Art. 870, for finding the logarithms of numbers which are not in the tables, and conversely.

Inasmuch as $n+1 = n\left(1+\frac{1}{n}\right)$, we find

$$\log(n+1) = \log n + \log\left(1+\frac{1}{n}\right),$$

and therefore

$$\log(n+1) - \log n = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \&c.:$$

and, if n be large, $\frac{1}{n}$ may be taken as an *approximate* value of this difference: thus, if $n = 10000$, $\frac{1}{2n^2}$ will have no significant digit in the first 8 places of decimals.

Again, since $(n+x) = n\left(1+\frac{x}{n}\right)$, we find

$$\log(n+x) = \log n + \log\left(1+\frac{x}{n}\right),$$

and therefore

$$\log(n+x) - \log n = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \ldots$$

and if x be small compared with n, then $\frac{x}{n}$ will be the *approximate* value of the difference of $\log(n+x)$ and $\log n$. If therefore $\Delta = \frac{1}{n}$ be the approximate value of the difference

$$\log(n+1) - \log n,$$

we shall find that $x\Delta$ will express the approximate value of the difference

$$\log(n+x) - \log n.$$

We conclude, therefore, that, if n be very large compared with x, the difference of the logarithms of $n+x$ and n will be nearly proportional to the difference of the numbers $n+x$ and x (Art. 869): and that if the difference of $\log(n+1)$ and $\log n$ be given, the difference of $\log(n+x)$ and $\log n$, when x is small compared with n, may always be found approximately by a simple proportion.

If the logarithms under consideration be tabular and not Napierian, then

$$\log(n+1) - \log n = \mu\left(\frac{1}{n} - \frac{1}{2n^2} + \&\text{c.}\right),$$

and

$$\log(n+x) - \log n = \mu\left(\frac{x}{n} - \frac{x^2}{2n^2} + \&\text{c.}\right),$$

where μ is the modulus of tabular logarithms (Art. 907): if Δ therefore be assumed to be $\frac{\mu}{n}$, then the formula

$$\log(n+x) = \log n + n\Delta$$

will furnish an approximate value of $\log(n+x)$: it will be at once seen that this proposition is the basis of the rules in Art. 870, to which we have before referred.

CHAPTER XXXVI.

ON THE LIMITS OF THE VALUES OF SERIES PROCEEDING ACCORDING TO ASCENDING OR DESCENDING POWERS OF A SYMBOL, WHICH IS CAPABLE OF INDEFINITE DIMINUTION OR INCREASE.

Limits of series.

915. THE following propositions relating to the limits of the values of series, proceeding according to ascending or descending powers of a symbol, which is capable of indefinite increase or diminution, will be found to be extremely useful in many enquiries, and more particularly in those which regard the application of Algebra to the properties of curvilinear figures. The complete theory of limits, properly so called, is only partially involved in them, and will be more expressly considered in a subsequent Chapter of this work.

In a geometric series.

916. If we make $x=\frac{\delta}{a+\delta}$, the sum of the geometric series

$$a+ax+ax^2+ax^3+\ldots$$

is equal to $a+\delta$.

For the sum of this series, when x is less than 1, is $\frac{a}{1-x}$ (Art. 432): and, if we make $x=\frac{\delta}{a+\delta}$, we get

$$s=\frac{a}{1-\frac{\delta}{a+\delta}}=\frac{a(a+\delta)}{a}=a+\delta.$$

It will follow, therefore, that a value of x, in such a series, may always be assigned, which will make its sum (s) differ from its first term a by a quantity less than any that can be assigned: for if δ be the quantity thus assigned, then any value of x which is less than $\frac{\delta}{a+\delta}$ $\left\{\text{say } x=\frac{\delta}{2(a+\delta)}\right\}$, will make s differ from a by a quantity less than δ.

917. A limit of a series or expression is a *fixed value* to which it approaches nearer than for any assignable difference, whilst a symbol upon which it is dependent is indefinitely diminished or indefinitely increased, but which it never attains whilst the value of that symbol is different from zero in one case or from infinity in the other. Definition of a limit.

918. It will follow therefore that the *limit* of the value of the series

$$a + ax + ax^2 + ax^3 + \ldots$$

is its first term or a: for it has been shewn that a value of x may be assigned, which will make its sum differ from a by a quantity less than any other that can be assigned: and it never attains to this limit, though it may approach indefinitely near to it, whilst x is different from *zero.* The limit of a geometric series proceeding according to the ascending powers of a symbol is its first term.

919. If in the series

$$a + \frac{a}{x} + \frac{a}{x^2} + \frac{a}{x^3} + \ldots$$

proceeding according to inverse powers of x, we make $x = \frac{a+\delta}{\delta}$, then its sum is equal to $a + \delta$. The limit of a geometric series proceeding according to inverse powers of a symbol.

For, if we make $y = \frac{1}{x}$, this series becomes

$$a + ay + ay^2 + ay^3 + \ldots$$

whose sum, when $y = \frac{\delta}{a+\delta}$, and therefore $x = \frac{a+\delta}{\delta}$, is $a + \delta$.

It will follow, therefore, that, in such a series, a value of x may always be assigned, which will make its sum (s) differ from its first term by a quantity less than any which may be assigned: for if δ be the quantity thus assigned, then any value of x greater than $\frac{a+\delta}{\delta}$ $\left\{\text{say } \frac{2(a+\delta)}{\delta}\right\}$ will make s less than $a + \delta$.

The *limit,* therefore, of the value of this series, when x is *indefinitely* (see Chap. XXXVII.) great, is its first term or a.

920. The terms of the geometric series

$$a + arx + ar^2x^2 + ar^3x^3 + \ldots \qquad (\alpha),$$

are severally either equal to or greater than the corresponding terms of the series Formation of a geometric series which is a superior limit to a given series pro-

$$a + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad (\beta),$$

ceeding according to powers of the same symbol.

if r be the greatest inverse ratio of any two consecutive coefficients.

In the first place, let $\frac{a_1}{a}$ be greater than $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, and all subsequent ratios of a similar kind: then we have $\frac{a_1}{a}=r$, and therefore $a_1=ar$: $\frac{a_2}{a}$ is less than r, and therefore a_2 is less than $a_1 r$, and therefore also less than ar^2, since $a_1=ar$: $\frac{a_3}{a_2}$ is less than r, and therefore a_3 is less than $a_2 r$, and therefore less than ar^3, since a_2 is less than ar^2: and similarly for all subsequent coefficients of the series: it follows, therefore, that the first and second terms of the geometric series (α) are equal to the first and second terms of the series (β), but that all the subsequent terms of the first series are severally greater than those corresponding to them in the second.

In the second place, let any other inverse ratio, such as $\frac{a_n}{a_{n-1}}$, and not the first $\frac{a_1}{a}$, be the greatest, and therefore equal to r: we have consequently $\frac{a_1}{a}$ less than r, and therefore a_1 less than ar: $\frac{a_2}{a}$ less than r, and therefore a_2 less than $a_1 r$, and *a fortiori* less than ar^2, since a_1 is less than ar: and so on, until we come to a_{n-1}, which is less than ar^{n-1}: the next ratio $\frac{a_n}{a_{n-1}}=r$, and therefore $a_n=a_{n-1}r$, which is less than ar^n, since a_{n-1} is less than ar^{n-1}: and similarly for all subsequent coefficients: it will follow, therefore, that the terms of the series (α), after the first, are severally greater than those corresponding to them in the series (β).

If the successive ratio, $\frac{a_1}{a}$, $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, as far as the n^{th} ratio $\frac{a_n}{a_{n-1}}$, be equal to each other, but greater than all those which follow them, then if $\frac{a_1}{a}=r$, the n first terms of the series, (α) and (β) are equal to each other: but all the subsequent terms of the series (α) are severally greater than those corresponding to them in the series (β).

We may assume the coefficients of the series (β) to be positive, and to increase perpetually as we recede from the first term: but the proposition which we have demonstrated above will be true *a fortiori* if the coefficients form a decreasing series, or if one or more of them become zero or negative.

It will follow generally, therefore, that if r be the greatest inverse ratio of any two consecutive coefficients, the terms of the geometric series (α) are severally equal to or greater than the corresponding terms of the series (β), and that consequently a value of x may always be determined, which will make the sum of the series (β) differ from its first term by a quantity less than any which may be assigned: thus, if $rx = \frac{\delta}{a+\delta}$, and therefore $x = \frac{\delta}{r(a+\delta)}$, the sum of the series (α) is $a + \delta$, and therefore the sum of the series (β) is less than $a + \delta$: and the *limit* of its value (Art. 917) is a, or the first term of the series.

Thus, in the series Examples.

$$1 \times 2 + 2 \times 3x + 3 \times 4x^2 + 4 \times 5x^3 + \dots \qquad (\beta),$$

the value of r or of the greatest inverse ratio of two consecutive coefficients is 3: and the terms of the geometric series

$$2 + 2 \times 3x + 2 \times 3^2x^2 + 2 \times 3^3x^3 + \dots \qquad (\alpha),$$

are severally equal to or greater than those of the series proposed (β): if $x = \frac{\delta}{3(2+\delta)}$, the sum of the terms of the series (β), will differ from its first term 2 by a quantity less than δ.

If the series proposed had been

$$1 + 1 \times 2x + 1 \times 2 \times 3x^2 + 1 \times 2 \times 3 \times 4x^3 + \dots$$

the ratio of the n^{th} to the $(n-1)^{\text{th}}$ coefficient is n, which increases indefinitely: the series is therefore infinite, if indefinitely continued, whatever be the value of x.

921. Proposition. If there be three quantities whose values are expressed by series proceeding according to powers of the same symbol, and if, for the same value of that symbol, the first be necessarily greater than the second and the second than the third; then, if the first and third series have the same first term or the same *limit*, the first term or limit of the second On the limit of a series whose first term is unknown, but which is included in value be-

tween two others which have the same limit.

series will be necessarily equal to it: it being assumed that the inverse ratio of any two consecutive coefficients of those series is always finite.

Thus, if the three series representing the three quantities be

$$a + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad (1),$$

$$b + b_1 x + b_2 x^2 + b_3 x^3 + \ldots \quad (2),$$

$$a + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \quad (3),$$

where the inverse ratio of any two consecutive coefficients is always finite, then values of x *exist* which will make their arithmetical values differ from their first terms by quantities less than any which may be assigned: let such values of the series be $a + \delta$, $b + \delta_1$, $a + \delta_2$: and since it is assumed that $a + \delta$ is greater than $b + \delta_1$, and $b + \delta_1$ greater than $a + \delta_2$, it will follow that the values of

$$a - b + \delta - \delta_1 \text{ and } b - a + \delta_1 - \delta_2$$

are arithmetical and positive: and, if possible, let us suppose $b = a + d$ or $a - d$: in the first case, the preceding expressions become

$$-d + \delta - \delta_1 \text{ and } d + \delta_1 - \delta_2,$$

and in the second

$$d + \delta - \delta_1 \text{ and } -d + \delta_1 - \delta_2:$$

and inasmuch as these expressions are necessarily positive, it will follow that, in the first case, $\delta - \delta_1$ is greater than d, or δ greater than $d + \delta$, which is contrary to the hypothesis, since d is supposed less than any quantity which may be assigned: and in the second case, $\delta_1 - \delta_2$ must be greater than d, or δ_1 greater than $d + \delta_2$, which is also contrary to the hypothesis, since δ_1 has been supposed less than any quantity which may be assigned: it follows, therefore, that b is necessarily equal to a, which is the proposition to be proved.

This is a proposition of very extensive application, inasmuch as it will very frequently happen that an expression, which is not capable of direct development into a series by the aid of assumed definitions or known theorems, may be shewn, by other considerations, to be included in value between expressions which admit of direct development, and which have therefore ascertainable limits; and it will follow that, if those limits be the same, the limit of the unknown expression or undeveloped series is necessarily the same likewise.

922. The correct notion of a limit is not easily formed, inasmuch as it is necessarily connected with a conception of a state of existence of magnitude, either in itself, or in some quantity upon which it is dependent, which is incapable of arithmetical or geometrical representation: and like all our notions, therefore, which ultimately involve considerations of zero or infinity (Chap. XXXVIII.), it is entirely negative: it is on this account that it becomes of the utmost importance that we should confine our attention exclusively to the definition (Art. 917), which we have given of it, and altogether disconnect it, like other definitions, from every consideration which is not essentially involved in it. Difficulties attending the conception of a limit.

A *limit*, in conformity with its definition, may be *zero*, but not *infinity*: for though we are incapable of conceiving *zero* as one of the successive states of existence of magnitude, we are capable of conceiving its existence in a state in which it differs from zero by a quantity less than any which may be assigned, and therefore when zero becomes, as it were, the fixed limit of the definition: but we are utterly incapable of conceiving the existence of a quantity which is not infinite, but which at the same time differs from infinity by a quantity less than any which may be assigned: and therefore, under no circumstances can infinity answer the conditions of a limit, which the definition assigns to it. A limit may be zero, but not infinity.

CHAPTER XXXVII.

ON THE SERIES AND EXPONENTIAL EXPRESSIONS FOR THE SINE AND COSINE OF AN ANGLE.

The series for the sine and cosine of x.

923. It has been already shewn (Art. 808) that the value of a in the exponential expressions

$$\frac{a^x + a^{-x}}{2} \quad \text{and} \quad \frac{a^x - a^{-x}}{2\sqrt{-1}}$$

is indeterminate, as far as it is dependent upon the definition of the sine and cosine of an angle only: we shall proceed, in the Articles which follow, to shew that it ceases to be indeterminate when the *measure* of angles ceases to be so.

The exponential series deduced in Chapter XXXV give us

$$a^x = 1 + kx + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} + \frac{k^4x^4}{1.2.3.4} + \dots$$

$$a^{-x} = 1 - kx + \frac{k^2x^2}{1.2} - \frac{k^3x^3}{1.2.3} + \frac{k^4x^4}{1.2.3.4} - \dots$$

where $k = \log a$: we thus find

$$\cos x = \frac{a^x + a^{-x}}{2} = 1 + \frac{k^2x^2}{1.2} + \frac{k^4x^4}{1.2.3.4} + \dots$$

$$\sin x = \frac{a^x - a^{-x}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}\left(kx + \frac{k^3x^3}{1.2.3} + \frac{k^5x^5}{1.2.3.4.5} + \dots\right).$$

If we further replace k by $c\sqrt{-1}$, these series become,

$$\cos x = 1 - \frac{c^2x^2}{1.2} + \frac{c^4x^4}{1.2.3.4} - \dots$$

$$\sin x = cx - \frac{c^3x^3}{1.2.3} + \frac{c^5x^5}{1.2.3.4.5} - \dots$$

The measure of an angle which is assumed in

924. It may be easily shewn that these series will satisfy the equation (Art. 758)

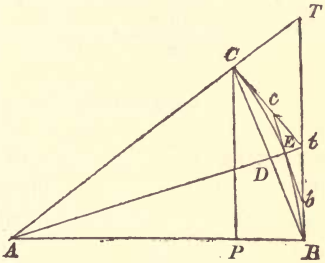
$$\sin^2 x + \cos^2 x = 1$$

when substituted in it, whatever be the value of c: but if we

assume, as we have already done (Art. 746), x or the measure of an angle, to be the ratio of the arc which subtends it to the radius of the circle in which it is taken, then it may be demonstrated that 1 is the only value of c, in the series for $\sin x$ and $\cos x$, which will satisfy the conditions to which it leads. Art. 746, determines the value of the symbol c which they involve.

For, if we assume x or the measure of the angle BAC to be $\frac{BEC}{AB}$, then since $\sin x = \frac{CP}{AB}$, we get

$$\frac{\sin x}{x} = \frac{CP}{AB} \times \frac{AB}{BEC} = \frac{CP}{BEC}:$$



and inasmuch as

$$\sin x = cx - \frac{c^3 x^3}{1.2.3} + \frac{c^5 x^5}{1.2.3.4.5} - \dots$$

we get

$$\frac{\sin x}{x} = c - \frac{c^3 x^2}{1.2.3} + \frac{c^5 x^4}{1.2.3.4.5} - \dots$$

where c, which is the *limit* of $\frac{\sin x}{x}$ (for it is the limit (Art. 918) of the series which is equivalent to it), is required to be determined.

For this purpose, we observe that the ratio $\frac{CP}{\text{chord } BC}$ is greater than $\frac{CP}{\text{arc } BEC}$ or $\frac{\sin x}{x}$, since the chord BC is less than the arc BEC: but $CP = AB \sin x$, and the chord $BC = 2AB \sin\frac{x}{2}$:

and therefore $\frac{CP}{\text{chord } BC} = \frac{\sin x}{2 \sin\frac{x}{2}} = \frac{2\cos\frac{x}{2}\sin\frac{x}{2}}{2\sin\frac{x}{2}}$ (Art. 775) $= \cos\frac{x}{2}$:

and again, inasmuch as the tangential line BT is greater than the arc BEC*, and since $\tan x = \frac{CP}{AP} = \frac{BT}{AB}$, we get $\frac{\sin x}{\tan x} = \cos x = \frac{CP}{BT}$ which is less than $\frac{CP}{\text{arc } BEC}$, and therefore less than $\frac{\sin x}{x}$.

* This may be easily shewn from geometrical considerations: if from the extremity C of the arc BEC, we draw Ct a tangent meeting BT in t, then we have $Bt = Ct$: but Tt is greater than Ct, being opposite to the greater angle TCt: therefore BT is greater than $Bt + Ct$: again, if from the middle point E of BEC, we draw the tangent cEb, meeting Ct in c and Bt in b, then we have $bt + ct$, greater

It appears, therefore, that $\frac{\sin x}{x}$ is interposed in value between $\cos\frac{x}{2}$ and $\cos x$: or, in other words, the quantities

$$\cos\frac{x}{2} = 1 - \frac{c^2x^2}{1.2.2^2} + \frac{c^4x^4}{1.2.3.4.2^4} - \dots$$

$$\frac{\sin x}{x} = c - \frac{c^3x^2}{1.2.3} + \frac{c^5x^4}{1.2.3.4.5} - \dots$$

$$\cos x = 1 - \frac{c^2x^2}{1.2} + \frac{c^4x^4}{1.2.3.4} - \dots$$

are arranged in the order of their magnitude; and since the first and third have a common limit, which is 1, the limit c of the second series is identical with it, and therefore equal to 1. (Art. 919).

greater than bc, and therefore $Bt + Ct$, and therefore *a fortiori* BT, greater than $Bb + bc + cC$: in a similar manner, if we bisect the arcs BE and CE, and from their middle points draw tangents meeting Bb, bc and Cc, then the sum of the tangents thus formed will be less than $Bb + bc + Cc$, and therefore *a fortiori* than BT: by continuing this process, we should increase the number and diminish the magnitude of these small tangents, until their sum, which is necessarily less than BT, shall differ from the arc BEC by a quantity or line less than any that may be assigned: or in other words, the arc BEC is necessarily less than BT: by a similar course of reasoning, the arc BEC may be shewn to be necessarily greater than the chord BC.

Again, if we take r for the radius of the circle, the series

$$r\tan x,\quad 2r\tan\frac{x}{2},\quad 2^2r\tan\frac{x}{2^2}, \dots 2^nr\tan\frac{x}{2^n}$$

will express BT, and the several sums of the successive circumscribing tangents: in a similar manner the series

$$2r\sin\frac{x}{2},\quad 2^2r\sin\frac{x}{2^2}, \dots 2^{n+1}r\sin\frac{x}{2^{n+1}}$$

will express the chord BC and the several sums of the successive inscribed chords: the ratio of the n^{th} term of the second series to the n^{th} term of the first is

$$\frac{2^{n+1}\sin\frac{x}{2^{n+1}}}{2^n\tan\frac{x}{2^n}} = \frac{\cos\frac{x}{2^n}}{\cos\frac{x}{2^{n+1}}},$$

and the limit of this value is 1: this is another mode of arriving at the conclusion in the text.

This relation of the chord, arc and circumscribing tangent, is one of fundamental importance in the application of Algebra to the theory of curves: it applies to the arcs of all curves of continuous curvature, as well as to arcs of the circle.

It consequently follows that

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots$$

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots$$

where x is the measure of an angle, which is formed by dividing the arc which subtends it at the centre, by the radius of the circle in which it is taken.

925. It appears, from the preceding investigation, that

Determinate exponential expressions for the sine and cosine.

$$c = 1 \text{ and } k = c\sqrt{-1} = \sqrt{-1}:$$

and, inasmuch (Art. 903) as

$$k = \log a = \sqrt{-1},$$

we find

$$a = e^{\sqrt{-1}}, \; a^x = e^{x\sqrt{-1}} \text{ and } a^{-x} = e^{-x\sqrt{-1}},$$

and therefore

$$\cos x = \frac{a^x + a^{-x}}{2} = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2},$$

$$\sin x = \frac{a^x - a^{-x}}{2\sqrt{-1}} = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}:$$

we thus obtain *determinate* and *explicit* exponential expressions for the sine and cosine, adapted to the commonly assumed measure of an angle.

926. Inasmuch as

Proof that $e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x$.

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2},$$

and

$$\sqrt{-1}\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2},$$

we get, by adding

$$\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}},$$

and by subtracting

$$\cos x - \sqrt{-1}\sin x = e^{-x\sqrt{-1}}.$$

927. It follows, therefore, since (Art. 811)

$$\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}\sqrt{-1} = e^{\frac{2r\pi}{n}\sqrt{-1}} = (1)^{\frac{1}{n}},$$

that the n roots of 1 are expressed by

$$e^0, \quad e^{\frac{2\pi}{n}\sqrt{-1}}, \quad e^{\frac{4\pi}{n}\sqrt{-1}}, \ldots\ldots e^{\frac{2(n-1)\pi}{n}},$$

and if the series be continued, the same values recur in the same order.

928. Again, since

$$\cos\frac{\pi}{2} + \sqrt{-1}\sin\frac{\pi}{2} = e^{\frac{\pi}{2}\sqrt{-1}},$$

or

$$\sqrt{-1} = e^{\frac{\pi}{2}\sqrt{-1}},$$

it follows that

$$\log\sqrt{-1} = \frac{\pi}{2}\sqrt{-1},$$

and therefore

$$\pi = \frac{2\log\sqrt{-1}}{\sqrt{-1}}.$$

Conclusions of this kind are *symbolical* only, and cannot be made the basis of calculation: and it is only in those cases in which the symbols or signs which such expressions involve, are capable of being interpreted, that we are enabled to conceive the nature of the connection between them and the specific numerical or other value which they may be shewn to denote.

Demoivre's formula.

929. Among many other important conclusions which follow immediately from the exponential expressions for the sine and cosine, it will be found that the formulæ of Demoivre, which are given in Art. 806, are immediately deducible from them.

For, if in

$$\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}}$$

we replace x successively by θ and by $n\theta$, we get

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}},$$

and

$$\cos n\theta + \sqrt{-1}\sin n\theta = e^{n\theta\sqrt{-1}} = (e^{\theta\sqrt{-1}})^n = (\cos\theta + \sqrt{-1}\sin\theta)^n:$$

and if in

$$\cos x - \sqrt{-1}\sin x = e^{-x\sqrt{-1}}$$

we replace x successively by θ and by $n\theta$, we get

$$\cos\theta - \sqrt{-1}\sin\theta = e^{-\theta\sqrt{-1}},$$

and

$$\cos n\theta - \sqrt{-1}\sin n\theta = e^{-n\theta\sqrt{-1}} = (e^{-\theta\sqrt{-1}})^n = (\cos\theta - \sqrt{-1}\sin\theta)^n.$$

930. The substitution of the exponential expressions for the sine, cosine or tangent of an angle, in the goniometrical formulæ in which they are involved, will enable us to resolve them, in many cases, into series, which are not only very remarkable in their form, but which admit of very useful applications.

Developement of the equation $\tan B = \frac{b\sin C}{a - b\cos C}$ by means of exponentials.

Thus, in a triangle, if a, b and the angle C be given, to determine A or B, we find

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin(B+C)}{\sin B} = \cos C + \frac{\cos B\sin C}{\sin B},$$

and therefore

$$\frac{\sin B}{\cos B} = \frac{b\sin C}{a - b\cos C}:$$

if we replace the sines and cosines by their equivalent exponential expressions, we get

$$\frac{e^{B\sqrt{-1}} - e^{-B\sqrt{-1}}}{e^{B\sqrt{-1}} + e^{-B\sqrt{-1}}} = \frac{b\,(e^{C\sqrt{-1}} - e^{-C\sqrt{-1}})}{2a - b\,(e^{C\sqrt{-1}} + e^{-C\sqrt{-1}})},$$

and therefore

$$\frac{e^{B\sqrt{-1}}}{e^{-B\sqrt{-1}}} = e^{2B\sqrt{-1}} = \frac{a - b e^{-C\sqrt{-1}}}{a - b e^{C\sqrt{-1}}} *.$$

If we take the logarithms of the two members of this equation, we get

$$\log e^{2B\sqrt{-1}} = \log(a - b e^{-C\sqrt{-1}}) - \log(a - b e^{C\sqrt{-1}})$$

$$= \log a + \log(1 - \frac{b}{a} e^{C\sqrt{-1}}) - \log a - \log(1 - \frac{b}{a} e^{C\sqrt{-1}}),$$

or

$$2B\sqrt{-1} = \frac{b}{a} e^{C\sqrt{-1}} + \frac{b^2}{2a^2} e^{2C\sqrt{-1}} + \frac{b^3}{3a^3} e^{3C\sqrt{-1}} + \dots$$

$$- \frac{b}{a} e^{-C\sqrt{-1}} - \frac{b^2}{2a^2} e^{-2C\sqrt{-1}} - \frac{b^3}{3a^3} e^{-3C\sqrt{-1}} - \dots,$$

* For if $\frac{x}{y} = \frac{c}{d}$, we get $\frac{x-y}{x+y} = \frac{c-d}{c+d}$, and conversely. Art. 296.

or

$$B = \frac{b}{a} \cdot \frac{e^{C\sqrt{-1}} - e^{-C\sqrt{-1}}}{2\sqrt{-1}} + \frac{b^2}{2a^2} \cdot \frac{e^{2C\sqrt{-1}} - e^{-2C\sqrt{-1}}}{2\sqrt{-1}} + \dots$$

$$= \frac{b}{a} \sin C + \frac{b^2}{2a^2} \sin 2C + \frac{b^3}{3a^3} \sin 3C + \dots$$

This series converges rapidly if b is small compared with a.

Developement of the equation $\tan x = m \tan \alpha$.

931. Similarly, in the equation

$$\tan x = m \tan \alpha,$$

which very often occurs in the applications of analysis, we get

$$\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{m\,(e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}})}{e^{\alpha\sqrt{-1}} + e^{-\alpha\sqrt{-1}}},$$

and therefore

$$e^{2x\sqrt{-1}} = e^{2\alpha\sqrt{-1}} \left(\frac{1 - \dfrac{m-1}{m+1} e^{-2\alpha\sqrt{-1}}}{1 - \dfrac{m-1}{m+1} e^{2\alpha\sqrt{-1}}} \right),$$

from whence we derive, as in the last Article,

$$x = \alpha + \frac{m-1}{m+1} \sin 2\alpha + \tfrac{1}{2} \left(\frac{m-1}{m+1}\right)^2 \sin 4\alpha + \text{\&c.}$$

If we replace m by $\tan\theta$, this series becomes, since

$$\frac{m-1}{m+1} = \tan(\theta - 45^0),$$

$$x = \alpha + \tan(\theta - 45^0) \sin 2\alpha + \tfrac{1}{2} (\tan\theta - 45^0)^2 \sin 4\alpha + \dots$$

Series for the measure of an angle in terms of its tangent.

932. Inasmuch as

$$\tan x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{\sqrt{-1}\,(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})},$$

we readily find

$$e^{2x\sqrt{-1}} = \frac{1 + \sqrt{-1} \tan x}{1 - \sqrt{-1} \tan x},$$

and therefore, taking the logarithms

$$2x\sqrt{-1} = 2\{\sqrt{-1} \tan x + \tfrac{1}{3}(\sqrt{-1} \tan x)^3 + \tfrac{1}{5}(\sqrt{-1} \tan x)^5 + \text{\&c.}\}$$

or

$$x = \tan x - \tfrac{1}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x - \dots$$

a series* remarkable for the simplicity of its form, by which the measure of an angle is expressed in terms of its tangent.

If we make x the measure of 45^0, and therefore $\tan 45^0 = 1$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \dots$$

a series which is very slowly convergent.

If we make x the measure of 30^0, and therefore $\tan x = \frac{1}{\sqrt{3}}$, we get

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \&c.\right),$$

a series which converges with great rapidity.

933. Among many other conclusions which are deducible from the series for the sine and cosine of an angle, it may be readily shewn that the sines of small angles are very nearly proportional to the angles themselves, a proposition whose truth we have assumed in a former Article (798): for if θ and θ' be the measures of two small angles, we find, neglecting those powers of θ which are higher than the third, that

The sines of small angles are nearly proportional to the angles themselves.

$$\frac{\sin\theta}{\sin\theta'} = \frac{\theta - \frac{\theta^3}{6}}{\theta' - \frac{\theta'^3}{6}} = \frac{\theta}{\theta'}\left\{1 + \frac{\theta'^2 - \theta^2}{6}\right\},$$

a ratio which becomes very nearly that of $\frac{\theta}{\theta'}$, when θ and θ' are very small, or differ very little from each other. Thus, let the angles measured by θ and θ' be $1'$ and $2'$ respectively: inasmuch as an arc, which is equal in length to the radius (Art. 747), contains $57.3^0 = 4438'$ nearly, we get $\theta = \frac{1}{4438}$ and $\theta' = \frac{2}{4438}$, and therefore

$$\frac{\sin 1'}{\sin 2'} = \tfrac{1}{2} \times 1.000000025$$
$$= \tfrac{1}{2} \text{ nearly.}$$

* This series was discovered by the celebrated James Gregory in 1771: if to 5 times the series in which $\tan x = \frac{1}{5}$ we add that in which $\tan x = \frac{1}{99}$, and subtract from their sum the series in which $\tan x = \frac{1}{70}$, the result will be the expression for $\frac{\pi}{4}$: it was by this method that the remarkable approximation, mentioned in the Note to Art. 747, was obtained.

Formulæ for the solution of triangles which involve the cosines of angles which differ little from zero, or 180°, or the sines of those which differ little from 90°.

934. Again, the ordinary formulæ for the solution of triangles sometimes give results, which are less accurate than those which are commonly obtained, when they involve the cosines of angles which differ little from zero or 180°, or the sines of those which differ little from 90°, inasmuch as, under such circumstances, their values change slowly for considerable changes in the value of the angle: the series, however, for the sines and cosines of angles, will enable us to deduce formulæ which are adapted to such extreme cases, and where the minutest changes of value in the quantities sought to be determined will become immediately sensible.

Thus, in a triangle, where A and B are very small and where C is very nearly 180°, we get (Art. 885), if $\pi - \theta$ be the measure of C,

$$\begin{aligned} c^2 &= a^2 + b^2 + 2ab\cos\theta \\ &= a^2 + b^2 + 2ab\left(1 - \frac{\theta^2}{2}\right) \\ &= (a+b)^2 - ab\theta^2, \end{aligned}$$

omitting all terms of the series for $\cos\theta$ beyond the second, as being too small to affect the result within the limits of the recorded places of figures; by extracting the square root, we get

$$c = (a+b)\left\{1 - \frac{ab\theta^2}{2(a+b)^2}\right\} \text{ nearly.}$$

Thus, if $C = 179^\circ$, $a = 20$, $b = 16$, we find $\theta = \frac{1}{57.3}$ nearly, and

$$\begin{aligned} c &= 36\left\{1 - \frac{10}{81(57.3)^2}\right\} \\ &= 35.999424 \text{ nearly.} \end{aligned}$$

Again, if α and β be the measures of A and B*, we get

$$\sin A = \alpha - \frac{\alpha^3}{6} \text{ nearly,}$$

$$\sin B = \beta - \frac{\beta^3}{6} \ldots$$

$$\sin C = \sin(A+B) = \alpha + \beta - \frac{(\alpha+\beta)^3}{6} \ldots$$

* When we speak of the sines and cosines of angles, it is indifferent whether the angles are expressed by their measures, or by degrees and minutes: but it should always be kept in mind that the *numerical* values of the measusres of angles enter essentially into the series which we are now considering.

Consequently

$$a = \frac{c \sin A}{\sin C} = \frac{c\left(\alpha - \frac{\alpha^3}{6}\right)}{\alpha + \beta - \frac{(\alpha+\beta)^3}{6}}$$

$$= \frac{c\,\alpha}{\alpha+\beta}\left\{1 - \frac{\alpha^2}{6} + \frac{(\alpha+\beta)^2}{6}\right\}$$

$$= \frac{c\alpha}{\alpha+\beta}\left(1 + \frac{2\alpha\beta+\beta^2}{6}\right),$$

$$b = \frac{c\beta}{\alpha+\beta}\left(1 + \frac{2\alpha\beta+\alpha^2}{6}\right)^*.$$

* If we add these expressions together, we find

$$a + b = c + \tfrac{1}{2}\,c\alpha\beta \text{ or } c = \frac{a+b}{1+\frac{1}{2}\alpha\beta},$$

a very convenient and very accurate formula for expressing one side of a triangle in terms of the two others and of the measures of the adjacent angles, provided those angles are small.

CHAPTER XXXVIII.

ON THE RELATIONS OF *ZERO* AND *INFINITY*.

Meaning of the terms *infinite* and *indefinite*.

935. THE terms *infinite* and *indefinite* are frequently used indiscriminately by mathematical writers, though, if due regard was paid to propriety of language, they should be distinguished from each other: they are *negative* terms whose meaning must be defined by that of the terms *finite* and *definite*, which are respectively opposed to them.

A *finite* number, a *finite* line, a *finite* space, a *finite* time, would denote any number, line, space, or time, which is either assigned or assignable: whilst the term *definite* could properly be applied to such of those quantities only as were already assigned or determined: in other words, the term *finite* is more comprehensive than *definite*, being limited only by the power possessed by the mind of conceiving the relations which the magnitudes, to which it is applied, bear to other magnitudes of the same kind.

An *infinite* number, an *infinite* line, an *infinite* space, an *infinite* time bear no conceivable or expressible relation to a *finite* number, a *finite* line, a *finite* space, or a *finite* time: the term *indefinite*, properly speaking, when applied to these quantities, would imply nothing more than that they were not determined or not assignable.

Infinity and *zero*.

936. Magnitudes may be *infinitely* small as well as *infinitely* great, and the abstract term *infinity* should be, properly speaking, equally applicable to both, though it is confined, by the usage of language, exclusively to the latter, whilst the term *zero* is exclusively applied to the former: the general term *infinity* is superseded by the specific terms *immensity* and *eternity* in the case of *space* and *time**.

* The phrase *for ever*, though properly expressing *infinite* duration of time, is commonly applied to denote *infinite* repetition as well as *infinite* time: thus the processes which never terminate are said to be continued for ever: the terms

Symbols of infinity and zero.

937. The symbol ∞ is used to denote magnitudes which are *infinitely* great, in the same manner that the symbol 0 is used to denote those which are *infinitely* small: they are connected by the equation $\frac{a}{0} = \infty$, and $\frac{a}{\infty} = 0$, where a may be a *finite* though *indeterminate* magnitude: in the first case, we consider ∞ as the quotient of the division of a by 0: in the second, we consider 0 as the quotient of a divided by ∞; such results may be interpreted by considering the *dividend* as the product of the *divisor* and quotient: thus, there is no *finite* number which, when multiplied into *zero* or an infinitely small number (fractional or decimal) will produce a finite product: there is no *finite* line, which multiplied into *zero* or an infinitely small line, will produce a *finite* area, and similarly in all other cases.

Different orders of infinities denoted by the same symbol.

938. The product of ∞ into 0, or of an infinitely great and an infinitely small number, line, or other magnitude, in the sense which we have attached to those symbols and terms, *may* produce a finite result, but it does not follow that it *must* do so: the equation $\frac{a}{0} = \infty$ is universally true, when a is finite: but the same equation is also true, when a is infinite, and therefore replaced by ∞: in other words $\frac{\infty}{0} = \infty$: in such a case, the infinity denoted by the symbol ∞, on one side of the equation, is said to be infinitely greater than the infinity denoted by the same symbol on the other: for one is equivalent to $\frac{a}{0}$, and the other to $\frac{\infty}{0}$, and the ratio of the second to the first, or $\frac{\infty}{0} \times \frac{0}{a}$ $= \frac{\infty}{a} = \infty$ or infinity.

The relation of the infinities denoted by the same symbol ∞ may become definite when the circumstances of their origin are known.

939. The mind is as incapable of conceiving the relation of different orders of infinities as it is of conceiving infinity itself, and it is only when the relation between them is the necessary result of symbolical language, and of those general

of a series which are said to be continued *in infinitum*, are also said to be continued for ever: but it should be observed, that the notion of infinity of time is closely associated in the mind with all our notions of indefinite repetition.

laws of their combination, which the rules of Algebra impose upon them, that they can become the proper object of our reasonings: for the same symbol ∞ is used equally to denote all magnitudes which are infinitely great, and the same symbol 0 to denote all magnitudes which are infinitely small: but if the symbols ∞ or 0 were used, in a course of operations as ordinary symbols, when the circumstances of their usage shewed a common origin, and therefore indicated something beyond a mere symbolical identity, we must adopt the results of operations upon them, whether finite or not, in the same manner as any other necessary results of Algebra: thus, if the symbols ∞ and ∞ denoted the same infinite magnitude (such as $\frac{2}{1-x}$, when $x=1$), the relation of $a \times \infty$ and $b \times \infty$, or $\frac{a \times \infty}{b \times \infty}$ would be equally $\frac{a}{b}$, as if ∞, in this ratio, had been replaced by an ordinary symbol of algebra: but, if the course of our reasonings should call our attention to $a \times \infty$ and $b \times \infty$, as simply denoting two infinite magnitudes, without any reference to the relation which the circumstances, in which they originated, may make them bear to each other, we might properly represent $a \times \infty$ and $b \times \infty$ by the common symbol ∞, and the relation between $a \times \infty$ and $b \times \infty$ would become altogether indeterminate.

Example. But though the symbol ∞ might not have the same origin in the expressions $a \times \infty$ and $b \times \infty$, yet if the precise symbolical conditions of their origin in both cases were known, the indetermination of their relation to each other might be removed: thus, if the symbol ∞ in $a \times \infty$ originated in the expression $\frac{2}{1-x}$ when $x=1$, and the symbol ∞ in $b \times \infty$ originated in the expression $\frac{2}{1-x^2}$ when $x=1$, we should find

$$\frac{a \times \infty}{b \times \infty} = \frac{a \times \dfrac{2}{1-x}}{b \times \dfrac{2}{1-x^2}}$$

$$= \frac{a \times (1-x^2)}{b \times (1-x)} \text{ (when } x=1) = \frac{a \times 0}{b \times 0}:$$

but inasmuch as it appears that

$$\frac{a \times (1-x^2)}{b \times (1-x)} = \frac{a(1+x)}{b}$$

for all values of x whatsoever, and therefore when $x=1$, it will follow that, under these circumstances

$$\frac{a \times \infty}{b \times \infty} = \frac{2a}{b},$$

and the relation of the infinite magnitudes represented, in the numerator and denominator of this expression, by the common symbol ∞, or by the common symbol 0, under another and equivalent form, is necessarily that of 2 to 1.

940. It may be useful to illustrate, by some other examples, the origin of the *indetermination* which exists, in certain cases in expressions which involve the common symbol 0 or the common symbol ∞.

Further exemplification of the origin of different orders of zeros and infinities.

The sum (s) of the series

$$ax + ax^2 + ax^3 + \ldots$$

and the value of x, are denoted by the common symbol 0, when x is zero: but inasmuch as

$$\frac{s}{x} = a + ax + ax^2 + \ldots$$

whose value is a when x is zero, it will follow, that if we denoted the first *zero* by $0'$ and the second *zero* by 0, we should find

$$\frac{0'}{0} = a,$$

or, in other words, a definite relation would thus be shewn to exist between the zeros denoted in this particular instance by $0'$ and 0: but all traces of this relation would be obliterated if $0'$ and 0 were replaced, as is usual, by the common symbol 0 as the common and only representative of *zero*, and if no reference was made to the circumstances of their origin.

Again, the sum (s) of the series

$$ax^2 + ax^3 + ax^4 + \ldots$$

is *zero*, when x is zero: if we divide s by x, we get

$$\frac{s}{x} = ax + ax^2 + ax^3 + \ldots$$

which is *zero* when x is zero: if we again divide $\frac{s}{x}$ by x, we get

$$\frac{s}{x^2} = a + ax + ax^2 + \ldots$$

whose value is a, when x is *zero*: if we denote the *zeros* corresponding to s and to $\frac{s}{x}$, when x is *zero*, by $0''$ and $0'$, and the *zero* corresponding to x by 0, we shall find $\frac{0''}{0} = 0'$, and therefore $\frac{0''}{0'} = 0$, whilst $\frac{0'}{0} = a$: in other words, the relation between the two first *zeros* is *zero*, but between the two last is a: but if we had denoted the three successive zeros $0''$, $0'$ and 0 by the common symbol 0, and had suppressed the consideration of all reference to their origin, their relations to each other would have been altogether indeterminate.

Again, the sum (s) of the series

$$\frac{a}{x} + a + ax + ax^2 + \ldots$$

is infinite when $x = 0$: and similarly the sum s' of the series

$$\frac{a}{x^2} + \frac{a}{x} + a + ax + ax^2 + \ldots$$

is also infinite, when $x = 0$: if we denote the first of these infinite magnitudes by ∞ and the second by ∞', we shall find, since $s' = \frac{s}{x}$, when $x = 0$,

$$\infty' = \frac{\infty}{0} \text{ and therefore } \frac{\infty'}{\infty} = \frac{1}{0},$$

or, in other words, the ratio of the infinite magnitudes denoted, in this particular instance, by ∞' and ∞, is also an infinite magnitude: if, however, we had denoted these infinite magnitudes, without reference to their origin, by the common symbol ∞, we should have lost all traces of the relations which existed amongst them.

Determinate values of expressions which become $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

941. It appears, therefore, from the preceding considerations, that expressions, which, under certain circumstances, become $\frac{0}{0}$ or $\frac{\infty}{\infty}$, may admit of determinate values, which are different

from 0 and ∞; and that this determination may always be effected when such expressions admit of equivalent forms, which do not become $\frac{0}{0}$ or $\frac{\infty}{\infty}$, under the same circumstances.

942. Thus, the obliteration of a common factor, which becomes 0 or ∞ for specific values of the symbols which it involves, may furnish an equivalent form of the original expression, which does not become $\frac{0}{0}$ or $\frac{\infty}{\infty}$, under the same circumstances.

Obliteration of common factors, which become 0 or ∞.

As an example, the fractional expression $\frac{x^3 - a^3}{x^2 - a^2}$ becomes $\frac{0}{0}$ when $x = a$: but if we obliterate the common factor $x - a$ (which becomes 0 when $x = a$) of its numerator and denominator, the fraction assumes the equivalent form

Examples.

$$\frac{a^2 + ax + x^2}{a + x},$$

which becomes $\frac{3a^2}{2a} = \frac{3a}{2}$, when $x = a$: its value, therefore, is no longer indeterminate.

Again, the expression

$$\frac{x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4}{x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4}$$

becomes $\frac{0}{0}$, when $x = a$: but if we obliterate the common factor $(x - a)^3$ of its numerator and denominator, the original fraction assumes the equivalent form $\frac{x + 4a}{x + 2a}$, which becomes $\frac{5}{3}$ when $x = a$.

943. But if *different* powers of a common factor exist in the numerator and denominator of the original fractional expression, the discovery and obliteration of what is common to both of them by the ordinary rule, will leave a residual power of this factor in one of them, which becomes 0 or ∞ for the specific values of the symbols which are under consideration: under such circumstances, the value of the fraction is either *zero* or *infinity*.

When different powers of a common factor exist in the numerator and denominator.

Thus, $a - 2x$ is found to be a common factor of the numerator and denominator of

$$\frac{3a^3 - 10a^2x + 4ax^2 - 8x^3}{a^2 - 4x^2},$$

which is thus reduced to the equivalent form

$$\frac{3a^2 - 4ax - 4x^2}{a + 2x}:$$

but inasmuch as $a - 2x$ still continues a factor of its numerator, the value of the fraction is *zero* when $x = \frac{a}{2}$: in a similar manner, the value of the fraction

$$\frac{3a^2 - 4ax - 4x^2}{a^3 - 2a^2x - 4ax^2 + 8x^3},$$

which becomes $\frac{0}{0}$ when $x = \frac{a}{2}$, and which also becomes $\frac{3a + 2x}{a^2 - 4x^2}$ when the common factor $a - 2x$ of its numerator and denominator is obliterated, will be found to be ∞ when $x = \frac{a}{2}$: for $a - 2x$ still continues to be a factor of its denominator.

When the expressions which become $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are irrational or involve exponential logarithmic or goniometrical quantities.

944. In many cases, however, the expressions which become $\frac{0}{0}$ or $\frac{\infty}{\infty}$, for specific values of the symbols which they involve, do not present themselves under a rational form, and the ordinary rule, therefore, for detecting their common factors ceases to be applicable. In the absence of the more prompt and certain methods of finding their determinate values, under such circumstances, whenever they exist, which will be given in a subsequent Chapter of this work, we may very frequently succeed in determining them by the following method.

Let the fraction $\frac{P}{Q}$ become $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x = a$: replace x by $a \pm h$, and develope P and Q (by the aid of the binomial theorem or other methods at our command), according to powers of h, whether integral or fractional: divide the series which result by any powers of h, which are common to all their terms, and subsequently making h *zero* or *infinity*, we shall find the determinate value of $\frac{P}{Q}$, which is sought for.

Examples.

Thus, the expression

$$\frac{\sqrt{(a - x)} + \sqrt{(ax - x^2)}}{\sqrt{(a^2 - x^2)}}$$

becomes $\frac{0}{0}$ when $x = a$: we replace x by $a - h$, and it becomes

$$\frac{\sqrt{h}+\sqrt{(ah-h^2)}}{\sqrt{(2ah-h^2)}}$$

$$=\frac{\sqrt{h}\{1+\sqrt{a}\,(1-\frac{h}{2a}-\frac{h^2}{8a^2}-\dots)\}}{\sqrt{2ah}\,(1-\frac{h}{4a}-\frac{h^2}{32a^2}-\dots)}=\frac{1+\sqrt{a}-\frac{h}{2a^{\frac{1}{2}}}-\dots}{\sqrt{2a}-\frac{h}{2\sqrt{2a}}-\dots}$$

which becomes

$$\frac{1+\sqrt{a}}{\sqrt{2a}}$$

when $h=0$*:

Let the expression be

$$\frac{a^n-x^n}{\log a-\log x},$$

which becomes $\frac{0}{0}$, when $x=a$: if we replace x by $a-h$, we get

$$\frac{a^n-(a-h)^n}{\log a-\log(a-h)}=\frac{na^{n-1}h-n(n-1)a^{n-2}\frac{h^2}{1\,.\,2}+\dots}{\frac{h}{a}+\frac{h^2}{2a^2}+\dots}$$

$$=\frac{na^{n-1}-n(n-1)a^{n-2}\frac{h}{1\,.\,2}+\dots}{\frac{1}{a}+\frac{h}{2a^2}+\dots}$$

$$=na^n,\text{ when } h=0 \text{ or } x=a.$$

Let the expression be

$$\frac{e^x-1-\log(1+x)}{x^2},$$

* It may be observed, that the common factor of a series of irrational expressions of the same order, or which are reducible to the same order, may be found, by finding the common factor of the rational expressions which are included under the same radical sign: the common factor, when subjected to the common radical sign, which is thus found, is the common factor of the irrational expressions under consideration.

Thus $a-x$ is the common factor of $a-x$, $ax-x^2$ and a^2-x^2: and therefore $\sqrt{(a-x)}$ is the common factor of the numerator and denominator of

$$\frac{\sqrt{(a-x)}+\sqrt{(ax-x^2)}}{\sqrt{(a^2-x^2)}},$$

which becomes, when reduced, $\frac{1+\sqrt{x}}{\sqrt{(a+x)}}$.

which becomes $\frac{0}{0}$ when $x=0$: if we develope e^x and $\log(1+x)$, we get

$$\frac{1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}-1-(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots)}{x^2}=1-\frac{x}{6}+\ldots$$

which becomes 1, when $x=0$.

Let the expression be

$$\frac{\sin n\theta\cos(n-1)\theta}{\sin\theta},$$

which becomes $\frac{0}{0}$, when $x=2\pi$ (Art. 844, Note); if we make $\theta=2\pi-h$, this expression becomes

$$\frac{-\sin nh\cos(n-1)h}{-\sin h}=\frac{(nh-\frac{n^3h^3}{1.2.3}+\&c.)\{1-\frac{(n-1)^2h^2}{1.2}+\&c.\}}{h-\frac{h^3}{1.2.3}}$$

$$=\frac{(n-\frac{n^3h^2}{1.2.3}+\ldots)\{1-\frac{(n-1)^2h^2}{1.2}+\ldots\}}{1-\frac{h^2}{1.2.3}+\ldots}$$

$=n$, when $h=0$*.

* An expression which assumes, for specific values of its symbols, the form $0\times\infty$, becomes $\frac{0}{0}$ by inverting the expression which becomes ∞: thus

$$\left(1-\frac{2x}{\pi}\right)\tan x$$

is $0\times\infty$, when $x=\frac{\pi}{2}$: but if we replace $\tan x$ by $\frac{1}{\cot x}$, it becomes

$$\frac{1-\frac{2x}{\pi}}{\cot x}=\frac{0}{0}, \text{ when } x=\frac{\pi}{2};$$

and its determinate value is $\frac{2}{\pi}$. In a similar manner, an expression such as

$$\frac{x}{x-1}-\frac{1}{\log x},$$

which becomes $\infty-\infty$, when $x=1$, assumes, by reducing the fractions to a common denominator, the equivalent form

$$\frac{x\log x-x+1}{(x-1)\log x},$$

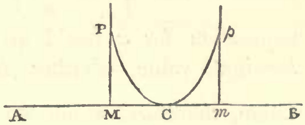
which becomes $\frac{0}{0}$ when $x=1$: its determinate value is $\frac{1}{2}$.

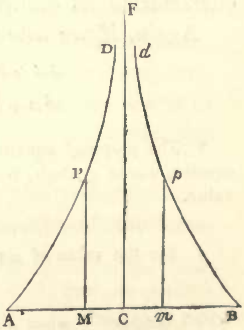
945. In the preceding examples, the values of $\frac{0}{0}$ and $\frac{\infty}{\infty}$, considered with reference to the expressions in which they originate, were *determinate, zero* or *infinity:* and if *zero* and *infinity* were considered as comprehended within the range of those possible existences which we are capable of contemplating equally with those which are the objects of calculation, we might regard them as *determinate* in all cases*: but in the examples which follow, $\frac{0}{0}$ or $\frac{\infty}{\infty}$ will be found to be symbols of *indeterminate* magnitude,

Cases in which $\frac{0}{0}$ or $\frac{\infty}{\infty}$ is the sign of indetermination.

* In the contemplation of continuous magnitude, we are accustomed to consider *zero* and *infinity* as equally included in the successive states of its existence, with those which are *finite* and *determinate:* thus, whilst the measure of an angle passes through all values between $\frac{\pi}{2} - h$ and $\frac{\pi}{2} + h$, including $\frac{\pi}{2}$, its cosine is said to pass through *zero*, and its tangent through *infinity* under the same circumstances, and we regard *zero* as the cosine, and *infinity* as the tangent of $\frac{\pi}{2}$, equally with the cosine or tangent of any other determinate angle.

It may be stated however as an objection to our considering such values as successive states of existence of the cosine and tangent, that they merely mark a point of transition in the condition of those successive magnitudes, from positive and negative, which may be considered as an interruption of their continuity : but it is easy to propose examples of a transition through *zero* and *infinity*, in which no such change of affection or sign takes place : thus if a curve PCp touch the line ACB in C, and if it be symmetrical with respect to it (such as the arc of a circle) on each side of the point C: then if, from successive points between P and p, we draw successive ordinates or perpendiculars from points in the curve, such as PM and pm to AB, they will pass through *zero* at C, and will be identical in magnitude and sign at points equidistant from C: in a similar manner, if the two curves APD and Bpd be symmetrical with respect to an *indefinite* line CF, which they never touch, but which they approach nearer than any assignable distance, (or in other words, if, in conformity with a phrase which will be afterwards explained, the indefinite line CF be a common *asymptote* to these curves), then for equal distances from C, the ordinates PM and pm will be identical in magnitude and sign, and will increase indefinitely as they approach C: in other words, PM may be said to pass through *infinity*, in passing through C, without any change in those conditions of its existence, which the signs of algebra are capable of expressing.





or the appropriate analytical forms by which the failure of the requisite conditions of determination is indicated.

Examples. Thus, in the equation

$$ax - b = a'x - b',$$

the ordinary process of solution gives us

$$x = \frac{b' - b}{a - a'}.$$

If we suppose $b' = b$, and a' not equal to a, we get

$$x = 0.$$

If we suppose $a' = a$, and b' greater than b, we get

$$x = \infty .$$

If we suppose $a' = a$ and $b' = b$, we get

$$x = \frac{0}{0}.$$

In the first case, 0, and in the second, ∞, are the only values of x, which will satisfy the proposed equation, in conformity with those symbolical properties of the symbols 0 and ∞, which distinguish them from ordinary symbols.

In the third case, all values of x will equally satisfy the conditions of the equation, which is therefore absolutely *indeterminate**: and $\frac{0}{0}$ $\left(\text{or } \frac{\infty}{\infty}\dagger\right)$ is the only form, which the general expression for x could assume, which would not give a *determinate* value, whether *finite, infinite,* or *zero:* the interpretation, therefore, which we have given of the expression $\frac{0}{0}$, in this case, is absolutely determined by the circumstances which characterize its occurrence.

Again, if we solve the simultaneous equations

$$ax + by = c \qquad (1),$$

$$a'x + b'y = c' \qquad (2),$$

* The proposed equation, under these circumstances, becomes an identical equation whose symbols, when not actually assigned, may assume every possible value.

† For the value of x may be expressed by $\dfrac{\left(\dfrac{1}{a' - a}\right)}{\left(\dfrac{1}{b' - b}\right)}$ equally with $\dfrac{b' - b}{a' - a}$, which becomes $\frac{\infty}{\infty}$ when $a' = a$ and $b' = b$.

by the ordinary process (Art. 395), we shall find

$$x=\frac{b'c-bc'}{ab'-a'b},$$

$$y=\frac{a'c-ac'}{a'b-ab'}.$$

If we suppose $a'=ma$, $b'=mb$, $c'=mc$, we shall get

$$x=\frac{mbc-mbc}{mab-mab}=\frac{0}{0},$$

$$y=\frac{mac-mac}{mab-mab}=\frac{0}{0},$$

which shew that x and y are *indeterminate* as far as these expressions are concerned: for the equations (1) and (2), under such circumstances, are not *independent* equations (Art. 391), the second being derived from the first, by multiplying each of its terms by the same symbol m, and it furnishes, therefore, no new condition for the determination of the symbols which it involves.

The values of x and y, in the equation

$$ax+by=c$$

are separately, though not simultaneously, capable of all values whatsoever, and are, therefore, when separately considered, absolutely indeterminate: but if we consider them as connected in the proposed equation with each other, the value of one of them determines the other, and one of them only can be considered as absolutely indeterminate with reference to the other.

946. In performing the operation of division and of the extraction of roots (Arts. 586, 650 and 693), when the operations do not terminate, and when the symbols are not arranged, as required in Arithmetic and Arithmetical Algebra, in the order of their magnitude, even when the expression is arithmetical in its original value, we obtain *diverging* series, whose terms are alternately positive and negative*, and whose values are therefore *indeterminate:* for, under such circumstances, no approximation is made to a determinate value by the aggregation of any number of their terms: and it has been observed that this *indetermination* is referrible to the neglect of that arrangement of the terms

Indeterminate series may result from expressions whose values are finite and arithmetical.

* Mr De Morgan, in a singularly original and learned Memoir in the Transactions of the Cambridge Philosophical Society (Vol. VIII. Part II.) classifies Divergent Series, as *alternating* or *progressing*, according as their terms are alternately positive and negative, or have all the same sign.

which are the subject of the operations performed, which the rules of Arithmetic and of Arithmetical Algebra prescribe: in such cases also the restitution of the arithmetical order of arrangement would convert *alternating* divergent, and therefore *indeterminate*, series, into others which were convergent and determinate, to whose values we can approximate as nearly as we choose, by the aggregation of a sufficient number of their terms.

It would be premature, in the present stage of our progress in analysis, to speculate generally upon the origin and character of *alternating and progressing divergent series.* Do *alternating* divergent series originate in expressions, which are always arithmetical in their value, though they may not be arithmetical in the circumstances and process of their developement? Do *progressing* divergent series always originate in expressions which are not arithmetical, either in their origin, in their arrangement, or in the process of their developement*? Will divergent series, either of one class or the other, when substituted, in algebraical operations, in place of the expressions in which they originate, or in which they are presumed to originate, give correct and equivalent results?

These questions, which are of fundamental importance in many of the most delicate and difficult applications of analysis, have given occasion to much dispute and controversy, and have not hitherto been satisfactorily settled. It will be prudent therefore for a student, whenever he is called upon to consider the infinite values, whether of series or of expressions under a definite form, to consider very carefully the circumstances in which they have their origin, before he draws any conclusion respecting the interpretation which they are capable of receiving.

The values of cos ∞ and sin ∞.

947. The symbol ∞ will sometimes present itself in expressions, which have periodical values, when the result will

* Thus $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \&c.$ and

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \&c.;$$

and when x is greater than 1, the first series is an *alternating*, and the second a *progressing* divergent series: but $\frac{1}{(1+x)^2}$ and $\frac{1}{(1-x)^2}$ are, under these circumstances, equally arithmetical in their value, though not in their arrangement of their terms: but can we consider $\frac{1}{(1-x)^2}$ as also arithmetical in its origin?

be apparently, if not absolutely, indeterminate, in consequence of our being unable to assign the position in the period to which it corresponds: thus, the values of $\cos\theta$ and $\sin\theta$ are periodical, their limits being 1 and -1, through the intervals of which they pass, whilst θ passes through intervals of value equal to 2π. What then is the value of $\cos\infty$ and $\sin\infty$ *? What is the value of $\tan\infty$ or $\cot\infty$? The answers to these questions are founded upon indirect considerations, to which we should never recur, when other resources are within our reach: they form points of transition between the demonstrated and acknowledged truths of deductive science and the less definite results of metaphysical speculation.

* See Mr De Morgan's Memoir above referred to, Sect. III: it is usual to assign 0 as their values, inasmuch as 0 is the *mean* of their successive periodic values: for it is equally probable that the value in question is a or $-a$: and if the value is assumed to be *unique*, 0 is the only value which the doctrine of chances, or similar considerations, would assign to it.

CHAPTER XXXIX.

ON THE DECOMPOSITION OF RATIONAL FRACTIONS WITH COMPOUND DENOMINATORS INTO PARTIAL FRACTIONS.

Rational algebraical fractions distinguished into two classes as proper and improper.

948. It is usual to distribute arithmetical fractions into two classes as *proper* and *improper* (Art. 95), according as the numerator is less or greater than the denominator: and the same denominations are extended to rational algebraical fractions according as the dimensions of the symbols in their numerators are lower or higher than those in their denominators: thus $\frac{3}{7}$ is a *proper* and $\frac{16}{7}$ is an *improper* numerical fraction: and similarly, $\frac{1+x}{1+x^2}$ is a *proper* and $\frac{1+x^4}{1-x^2}$ is an *improper* algebraical fraction.

Improper fractions are reducible to a rational expression and a proper fraction.

949. In the same manner, that *improper* numerical fractions are reducible, by the division of the numerator by the denominator, to an integer and a *proper* fraction, so likewise *improper* algebraical fractions are reducible by a similar process of division to an integral expression and a *proper* algebraical fraction: in this manner, $\frac{30}{13}$ is reduced to the mixed number $2\frac{4}{13}$, and similarly $\frac{1+x^4}{1+x^2}$ is reduced to $x^2-1+\frac{2}{1+x^2}$ where x^2-1 is an integral algebraical expression and $\frac{2}{1+x^2}$ a proper algebraical fraction: and similarly in all similar cases.

Example of the resolution of an algebraical fraction.

950. But the proper fraction, which is thus obtained, is farther resolvible into other and more simple fractions, when its denominator is resolvible into factors: thus suppose it was required to resolve the fraction

$$\frac{x+1}{x^2-7x+12}$$

into partial and more simple fractions, whose denominators are the factors $x-3$ and $x-4$ of its denominator

$$x^2-7x+12:$$

for this purpose, we should assume

$$\frac{x+1}{x^2-7x+12}=\frac{A}{x-3}+\frac{B}{x-4}$$

where A and B are unknown and are required to be determined.

If we add the partial fractions together, we get

$$\frac{x+1}{x^2-7x+12}=\frac{Ax-4A+Bx-3B}{x^2-7x+12},$$

and therefore

$$x+1=(A+B)x-(4A+3B).$$

Inasmuch as the value of x is *indeterminate*, this equation can only exist for all values of x, by supposing the corresponding terms of both its members to be severally *identical* with each other; we thus get

$$A+B=1,$$

$$4A+3B=-1,$$

and therefore $A=-4$, and $B=5$: we consequently find

$$\frac{x+1}{x^2-7x+12}=\frac{-4}{x-3}+\frac{5}{x-4}.$$

951. The principles which are involved in this process, though simple and obvious, are of considerable importance and of very extensive application. Principles which are involved in this process.

In the first place, the relation between $\dfrac{x+1}{x^2-7x+12}$ and $\dfrac{(A+B)x-(4A+3B)}{x^2-7x+12}$ is not merely one of *equality*, but also of *identity*, where x may have any value whatsoever, provided it is *simultaneous* in both: and we thus get as many conditions, or *equations*, as there are corresponding terms on each side, and therefore as many equations as there are unknown quantities A and B to be determined.

Identity and equality.

More generally, if any two series, proceeding according to the same law, such as

$$a + bx + cx^2 + dx^3 + \ldots\ldots(1)^*$$
$$\alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots\ldots(2)$$

be *identical* as well as *equal,* the corresponding terms of each series must be identical also, which will furnish as many equations as there are terms in each: for if not, the value of x can be no longer indeterminate, admitting of all values between *zero* and *infinity* inclusive: we have assumed the truth of this proposition in the investigation of the series for a^x in Art. 898.

If the members of the equation

$$x + 1 = (A + B)x - (4A + 3B)$$

were merely *equal* to each other and not *identical,* the symbols A and B would be indeterminate, and the value of x (as would appear by the ordinary solution of the equation) would be expressible in terms of them, and therefore be dependent upon them: it is the additional hypothesis of the identity of *form* as well as of the *equality* of value of the two members of this equation, which furnishes the equations for the determination of A and B, and leaves x indeterminate, as is the case with the symbols involved in any other identical equation.

General process for resolving rational fractions with decomposable denominators. When the factors of the denominator are possible and unequal.

952. The following method of resolving rational fractions with decomposable denominators, into an equivalent series of partial fractions, is general, and will serve to illustrate the preceding observations.

Let $\frac{M}{N}$ be a proper and rational fraction, and let $a + bx$ be a factor of N: if we make $N = (a + bx)Q$, we get

$$\frac{M}{N} = \frac{M}{(a + bx)Q} = \frac{A}{a + bx} + \frac{P}{Q},$$

and therefore

$$\frac{M}{Q} = A + \frac{P(a + bx)}{Q}.$$

* For if not, by subtracting the *equal* expressions (1) and (2) from each other, we get

$$0 = a - \alpha + (b - \beta)x + (c - \gamma)x^2 + (d - \delta)x^3 + \&c.,$$

where x can be neither zero nor infinity, unless $a - \alpha$ and the other coefficients are also zero.

If we now suppose M to become m and Q to become q, when $a+bx=0$ or $x=-\frac{b}{a}$, we shall find

$$A=\frac{m}{q}*.$$

Thus, let it be required to resolve Example.

$$\frac{x^2}{(x+1)(x+2)(x+3)}$$

into partial fractions.

Assume

$$\frac{x^2}{(x+1)(x+2)(x+3)}=\frac{A_1}{x+1}+\frac{A_2}{x+2}+\frac{A_3}{x+3}:$$

we thus get

$$\frac{M}{Q}=\frac{x^2}{(x+2)(x+3)}: x+1=0 \text{ and } x=-1: A_1=\frac{m}{q}=\frac{1}{(2-1)(3-1)}=\frac{1}{2}.$$

$$\frac{M}{Q'}=\frac{x^2}{(x+1)(x+3)}: x+2=0 \text{ and } x=-2: A_2=\frac{m}{q'}=\frac{4}{(1-2)(3-2)}=-4.$$

$$\frac{M}{Q''}=\frac{x^2}{(x+1)(x+2)}: x+3=0 \text{ and } x=-3: A_3=\frac{m}{q''}=\frac{9}{(1-3)(2-3)}=\frac{9}{2}.$$

Consequently

$$\frac{x^2}{(x+1)(x+2)(x+3)}=\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}.$$

953. If $(a+bx)^r$ be a factor of N, where r is greater than 1, it will be found, if we make $N=(a+bx)Q$, that $a+bx$ will still be a factor of Q, and consequently $\frac{P(a+bx)}{Q}$ will become $\frac{0}{0}$ and not necessarily 0, when $a+bx=0$: under such circumstances we make $N=(a+bx)^r Q$, and assume When some of the factors are possible and equal.

$$\frac{M}{N}=\frac{A_1}{(a+bx)^r}+\frac{A_2}{(a+bx)^{r-1}}+\ldots\frac{A_r}{A+bx}+\frac{P}{Q},$$

* It will be found that $P=\frac{0}{0}$, under the same circumstances: for it may be readily shewn that $P=\frac{M-AQ}{a+bx}$, and if $a+bx=0$, then $M-AQ=0$, otherwise P would be *infinite:* it follows therefore that $a+bx$ is also a factor of $M-AQ$, and if this factor be obliterated by division, the value of P will be found.

and therefore

$$\frac{M}{Q} = A_1 + A_2(a+bx) + \dots A_r(a+bx)^{r-1} + \frac{P(a+bx)^r}{Q}:$$

consequently, if we make $a+bx=0$, we get

$$A_1 = \frac{m}{q}.$$

Again, transposing A_1, we find

$$\frac{M}{Q} - A_1 = A_2(a+bx) + \dots A_r(a+bx)^{r-1} + \frac{P}{Q}(a+bx)^r,$$

both members of which equation are divisible by $a+bx$: if we make $M' = \frac{M - A_1 Q}{a+bx}$, we get

$$\frac{M'}{Q} = A_2 + A_3(a+bx) + \dots A_r(a+bx)^{r-2} + \frac{P}{Q}(a+bx)^{r-1},$$

which gives $A_2 = \frac{m'}{q}$.

In a similar manner, by making $M'' = \frac{M' - A_2 Q}{a+bx}$, we find $A_3 = \frac{m''}{q}$: and the continuation of this process will enable us successively to determine $A_4, A_5, \dots$ as far as A_r.

Example. Thus, suppose it was required to resolve

$$\frac{x^3+2x}{(x-2)^2(x+3)}$$

into a series of partial fractions.

Assume

$$\frac{x^3+2x}{(x-2)^3(x+3)} = \frac{A_1}{(x-2)^3} + \frac{A_2}{(x-2)^2} + \frac{A_3}{x-2} + \frac{B}{x+3},$$

which gives

$$\frac{M}{Q} = \frac{x^3+2x}{x+3} = A_1 + A_2(x-2) + A_3(x-2)^2 + B\frac{(x-2)^3}{x+3},$$

and therefore, making $x-2=0$ or $x=2$, we get

$$A_1 = \frac{m}{q} = \frac{12}{5}.$$

In the second place, we get

$$\frac{\frac{M}{Q}-A_1}{x-2}=\frac{x^3-\frac{2x}{5}-\frac{36}{5}}{(x+3)(x-2)}=\frac{x^2+2x+\frac{18}{5}}{x+3}=\frac{M'}{Q}:$$

and therefore, making $x=2$, we find

$$A_2=\frac{m'}{q}=\frac{58}{25}.$$

In the third place, we get

$$\frac{\frac{M'}{Q}-A_2}{x-2}=\frac{x^2+2x+\frac{18}{5}-\frac{58}{25}(x+3)}{(x+3)(x-2)}=\frac{x+\frac{42}{25}}{x+3}=\frac{M''}{Q}:$$

and therefore, making $x=2$, we find

$$A_3=\frac{m''}{q}=\frac{92}{125}.$$

Lastly, if we make $M=x^3+2x$ and $Q=(x-2)^3$, we find, by making $x+3=0$, or $x=-3$,

$$B=\frac{m}{q}=\frac{33}{125}.$$

Consequently,

$$\frac{x^3+2x}{(x-2)^3(x+3)}=\frac{12}{5(x-2)^3}+\frac{58}{25(x-2)^2}+\frac{92}{125(x-2)}+\frac{33}{125(x+3)}.$$

954. If one of the quadratic factors of N be of the form When some of the factors involve $\sqrt{-1}$.

$$x^2-2ax+a^2+\beta^2,$$

whose simple factors are $x-a-\beta\sqrt{-1}$, and $x-a+\beta\sqrt{-1}$, we collect the terms

$$\frac{A_1}{x-a-\beta\sqrt{-1}}+\frac{A_2}{x-a+\beta\sqrt{-1}}$$

into a single term of the form

$$\frac{Ax+B}{x^2-2ax+a^2+\beta^2},$$

and assume

$$\frac{M}{N}=\frac{Ax+B}{x^2-2ax+a^2+\beta^2}+\frac{A_3}{x-a_3}+\dots$$

If we make $N=(x^2-2ax+a^2+\beta^2)Q$ and replace x by $a+\beta\sqrt{-1}$, when M becomes $m+m_1\sqrt{-1}$ and Q becomes $q+q_1\sqrt{-1}$, we get

$$\frac{m+m_1\sqrt{-1}}{q+q_1\sqrt{-1}}=\frac{mq+m_1q_1}{q^2+q'^2}+\frac{(mq_1-m_1q)\sqrt{-1}}{q^2+q_1^2}\ *$$

$$=A(a+\beta\sqrt{-1})+B=Aa+B+A\beta\sqrt{-1}.$$

We thus find

$$A=\frac{(mq_1-m_1q)}{\beta(q^2+q_1^2)},\quad B=\frac{\beta(mq+m_1q_1)-a(mq_1-m_1q)}{\beta(q^2+q_1^2)}.$$

Thus, let us suppose

$$\frac{x}{(x-4)(x^2+4x+5)}=\frac{Ax+B}{x^2+4x+5}+\frac{C}{x-4}:$$

where $M=x$ and $Q=\dfrac{N}{x^2+4x+5}=x-4$. Replacing x by $-2-\sqrt{-1}$, we get $a=-2$, $\beta=-1$, $m=-2$, $m_1=-1$, $q=-6$, $q_1=-1$, and therefore

$$A=\frac{-4}{37}\text{ and }B=\frac{5}{37}.$$

If, in the second place, we make $M=x$, $Q=x^2+4x+5$ and $x=4$, we get

$$C=\frac{m}{q}=\frac{4}{37}:$$

we therefore find

$$\frac{x}{x^3-11x-20}=\frac{4}{37(x-4)}+\frac{-4x+5}{37(x^2+4x+5)}.$$

If there be a single factor of N of the form

$$(x^2-2ax+a^2+\beta^2)^r=u^r$$

we assume

$$\frac{M}{N}=\frac{A_1x+B_1}{u^r}+\frac{A_2x+B_2}{u^{r-1}}+\dots\frac{A_rx+B_r}{u}+\frac{P}{Q},$$

* For $$\frac{m+m_1\sqrt{-1}}{q+q_1\sqrt{-1}}=\frac{(m+m_1\sqrt{-1})(q-q_1\sqrt{-1})}{(q+q_1\sqrt{-1})(q-q_1\sqrt{-1})}$$

$$=\frac{mq+m_1q_1+(mq_1-m_1q)\sqrt{-1}}{q^2+q_1^2}.$$

which gives

$$\frac{M}{Q} = A_1 x + B_1 + (A_2 x + B_2)\, u - \ldots + (A_r x + B_r)\, u^{r-1} + \frac{P u^r}{Q};$$

we then determine A_1 and B_1 as in the last case, and subsequently making

$$\frac{M - (A_1 x + B_1)\, Q}{u} = M', \text{ we get}$$

$$\frac{M'}{Q} = A_2 x + B_2 + (A_3 x + B_3)\, u + \ldots (A_r x + B_r)\, u^{r-2} + \frac{P u^{r-1}}{Q}:$$

we then determine A_2 and B_2 as before, and proceed in the same manner as far as A_r and B_r.

By the application of this process we shall be enabled to resolve the fraction

$$\frac{x^2}{(x^2+4)^2\,(x-5)}$$

into the equivalent series of partial fractions

$$\frac{4\,(x+5)}{29\,(x^2+4)^2} - \frac{25\,(x+5)}{29^2\,(x^2+4)} + \frac{25}{29^2\,(x-5)}.$$

CHAPTER XL.

ON THE ASSUMPTION AND DETERMINATION OF SERIES.

Direct and indirect methods of determining equivalent forms.

955. THE methods which are employed for the determination of equivalent forms, may be considered as constituting two great classes, according as they are *direct* or *indirect*: *direct* methods are those in which the transition from the primitive to the equivalent form, is effected by means of defined or definable operations, such as multiplication, division, the raising of powers, and the extraction of roots; whilst *indirect* methods are generally resorted to, when we are unable to express in words or otherwise the nature of the operations, which connect the primitive forms with those which are equivalent to them: but, under such circumstances, the primitive form will commonly furnish the conditions which the derived or secondary form must satisfy, in order that its equivalence may be determined.

Use of indeterminate coefficients in direct developements.

956. Though the employment of *indeterminate* coefficients (Art. 898) whose use, in one important application, we have exemplified in the last Chapter, properly belongs to the second of these methods, yet it is likewise applicable to cases of direct developement, by a simple change, from direct to inverse, in the character of the operations which are required to be performed: the following examples will be sufficient to explain our meaning.

Example.

The series corresponding to the fraction

$$\frac{A + Bx}{a + bx + cx^2}$$

may be found by the actual division of the numerator by the denominator: but as it is readily seen that it proceeds by ascending powers of x, we may begin by assuming

$$\frac{A + Bx}{a + bx + cx^2} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$$

where $A_0, A_1, A_2, A_3 \ldots$ are *indeterminate** coefficients, whose values are required to be assigned: for this purpose we *multiply* both sides of this equation, instead of *dividing* $A + Bx$, by $a + bx + cx^2$, which gives us

$$\begin{aligned} A + Bx = A_0 a + A_1 a x &+ A_2 a x^2 + A_3 a x^3 + A_4 a x^4 + \ldots \\ &+ A_0 b x + A_1 b x^2 + A_2 b x^3 + A_3 b x^4 + \ldots \\ &\qquad + A_0 c x^2 + A_1 c x^3 + A_0 c x^4 + \ldots \end{aligned}$$

In order that these results may be *identical* as well as *equal*, we must have the corresponding terms identical with each other: we thus get

$$\begin{aligned} &A_0 a = A, \\ &A_1 a + A_0 b = B, \\ &A_2 a + A_1 b + A_0 c = 0, \\ &A_3 a + A_2 b + A_1 c = 0 \dagger, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The successive solution of these equations, in which A_0, A_1, A_2, $A_3 \ldots$ are the unknown quantities, will give us

$$\begin{aligned} A_0 &= \frac{A}{a}, \\ A_1 &= \frac{B}{a} - \frac{Ab}{a^2}, \\ A_2 &= -\frac{Bb}{a^2} + \frac{Ab^2}{a^3} - \frac{Ac}{a^2}, \\ A_3 &= \frac{Bb^2}{a^3} - \frac{Ab^3}{a^4} + \frac{Abc}{a^5}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The law of formation of the two last coefficients A_2 and A_3

* The term *indeterminate*, which is generally used in such cases, might be more properly replaced by *undetermined* or *unknown*: there is generally nothing indefinite in its character when thus applied.

† Such series, resulting from the ordinary process of division in algebra, are sometimes called *recurring* series, and the multipliers employed in the formation of the successive coefficients, *connected with their proper signs*, constitute what is called the *scale of relation*: thus the *scale of relation* in the series under consideration is $-\frac{b}{a} - \frac{c}{a}$, and in the numerical example which follows, it is -2 -3. Scale of relation.

will extend to all those which follow, and the series may thus be continued at pleasure.

Thus, if $A=1$, $B=3$, $a=1$, $b=2$, $c=3$, we find

$$A_0=1,\quad A_1=1,\quad A_2=-5,\quad A_3=7, \text{ \&c.},$$

and therefore

$$\frac{1+3x}{1+2x+3x^2}=1+x-5x^2+7x^3-11x^4+\dots$$

Again, assuming

$$\frac{\sin a-e\sin(a-b)}{1-2e\cos b+e^2}=A_0\sin a+eA_1\sin(a+b)+e^2A_2\sin(a+2b)+\dots$$

and multiplying both sides by $1-2e\cos b+e^2$, we get

$$\begin{aligned}\sin a-e\sin(a-b)=A_0\sin a+eA_1\sin(a+b)+e^2A_2\sin(a+2b)+\dots\\ -2eA_0\sin a\cos b-2e^2A_1\sin(a+b)\cos b-\dots\\ +e^2A_0\sin a.\end{aligned}$$

Equating the coefficients of corresponding terms, we find

$$A_0\sin a=\sin a;$$

and therefore $A_0=1$:

$$A_1\sin(a+b)-2\sin a\cos b=-\sin(a-b),$$

$$\text{or}\quad A_1\sin(a+b)=2\sin a\cos b-\sin(a-b)=\sin(a+b);$$

and therefore $A_1=1$:

$$A_2\sin(a+2b)-2\sin(a+b)\cos b+\sin a=0,$$

$$\text{or}\quad A_2\sin(a+2b)=2\sin(a+b)\cos b-\sin a=\sin(a+2b);$$

and therefore $A_2=1$: and similarly for all subsequent terms: we thus find

$$\frac{\sin a-e\sin(a-b)}{1-2e\cos b+e^2}=\sin a+e\sin(a+b)+e^2\sin(a+2b)+\dots *$$

a result which is equally deducible for all values of e.

* This series is a *recurring* series, whose scale of relation is $2e\cos b-e^2$: for

$$e^2\sin(a+2b)=2e\cos b\times e\sin(a+b)-e^2.\sin a,$$

$$e^3\sin(a+3b)=2e\cos b\times e\sin(a+2b)-e^2\sin(a+b).$$

If we call s the sum of this series indefinitely continued, we find

$$s=\sin a+e\sin(a+b)+2e\cos b(s-\sin a)-e^2s,$$

and therefore

$$s(1-2e\cos b+e^2)=\sin a-e\sin(a-b),$$

957. The form of the series in the last Example is peculiar, and such as could not have been assumed arbitrarily, or without some previous knowledge of the relation existing between the generating *expression** and its developement: but in the absence of such a knowledge, the process under consideration might appear likely to lead sometimes to the assumption of series, whether they possessed a necessary existence in the form assigned to them, or not: or, in other words, whether the operations which, on one side of the equation are indicated and not performed, would conduct us when they are performed to such a series or not: but a very little consideration will be sufficient to shew that the process employed for the determination of the *indeterminate* members of the assumed series, involves the essential conditions upon which the equivalence of the resulting expressions depends: and that the failure, in the determination of those coefficients that are erroneously or unnecessarily assumed, or their entire disappearance from the final result, would furnish the requisite correction of the first assumption, or, in other words, would be considered as the proper indication of the non-existence of the assumed equivalent expression or series, or at least of some one or more of its terms, under the form which was assigned to them.

This process is not vitiated by an erroneous assumption of the form of the series.

Thus, if we should assume

$$\frac{1}{1+x} = \frac{A_{-1}}{x} + A_0 + A_1 x + A_2 x^2 + \ldots$$

Examples of series assumed, with superfluous terms.

and proceed to the determination of the indeterminate coefficients, by the comparison of the terms of the two identical results deduced in the ordinary way, we should find $A_{-1} = 0$; and it would appear therefore that the assumption, in this case of a term of a form, which has no existence in the equivalent or resulting series, would lead to no error in the result obtained: again, if we should assume

$$\frac{1-x^2}{1+x^2-x^4} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + \ldots$$

$$\text{and } s = \frac{\sin a - e\sin(a-b)}{1 - 2e\cos b + e^2},$$

where s is the expression which generates the series in the text: it is only when e is less than 1 that this reasoning can be considered as strictly correct, unless a term, infinitely distant from the beginning, may be equally neglected, whether it be infinitely small or infinitely great.

* Instead of *generating expression*, it is more common to use the term *generating function*.

and proceed, in a similar manner, to the determination of the several coefficients of the series, we should find A_1, A_3, A_5, and all other coefficients of the terms which involve odd powers of x, severally equal to zero, and the correct result

$$1 - 2x^2 + 3x^4 - 5x^6 + 8x^8 - \ldots$$

would be determined in the same manner as if we had commenced with the assumption of the series

$$A_0 + A_2x^2 + A_4x^4 + A_6x^6 + \ldots$$

in which all superfluous terms were omitted.

Considerations which guide us in the assumption of series.

958. Though the correct assumption, therefore, of the form of the equivalent series in the first instance, is not absolutely essential to the correct determination of the series itself, yet it will always, more or less, tend to simplify the process for that purpose, by lessening the number of quantities to be determined, and by not encumbering the equations, whose solutions are required, with unnecessary symbols: it is for this and other reasons, that it becomes important, in all such researches, to avail ourselves of any considerations which may serve to guide us in the form which must be assumed for the series.

The general principle of the method of *indeterminate* coefficients, as we have already seen, (Art. 898) is to deduce two expressions or series, which, from the nature of the process by which they are obtained, are identical with each other: and this identity will be found generally to exist for *all* values whatsoever of the symbols which they involve: but the primitive expression, and the assumed series which is equivalent to it, may not always admit of the same range of values with the identical expressions to which the developement of the series may lead; thus some of these values may change the form of the primitive expression by making factors or terms of it 0 or ∞, when the assumed series may cease to be equivalent: whilst others may make the assumed series divergent, when the arithmetical equality between it and the expression from which it is derived, will cease to exist, though their algebraical equivalence may, in a certain sense, be still said to continue *. If the ex-

* It would be premature, at this stage of a student's progress, to enter upon the formal discussion of the much disputed question of the possible algebraical equivalence

pression, whose developement is required, be finite, or zero, when the symbol or combination of symbols, according to which the series is required to be arranged, is zero, then in neither case can its negative powers present themselves in one or more terms of its developement; and it will be found that its first term is the finite value thus determined in one case, and zero in the other*: but if, under the same circumstances, the expression becomes infinite, then there is one term, at least, of the resulting series which involves its negative power.

Again, if the sign of the expression, in which a series originates, changes or does not change its sign, for a change of

equivalence of a finite expression and a diverging series, of course confining the application of the term *equivalence* to its use as a factor in multiplication or in any other known algebraical operation: but it may be asked, if the product of such a series with a given expression, or the result of any other algebraical operation performed with it, is a definite symbolical result, when the series, which is thus employed, though general in its form, is assumed to be convergent, why should it not *necessarily* produce the same symbolical result, when, without any change of its form, it is assumed to be divergent? It is in fact at least as difficult to reject divergent series, when viewed either as the results or the subjects of operations, as it is to admit them when viewed as the representatives of arithmetical or other magnitudes.

Thus the expression or *generating function* (Art. 956)

$$\frac{\sin a - e \sin (a-b)}{1 - 2e\cos b + e^2}$$

will generate the series

$$\sin a + e\sin(a+b) + e^2 \sin(a+2b) + \ldots$$

whatever be the value of e, whether less, equal to, or greater than 1; and in the two first cases, it will likewise express the arithmetical value or *sum* of the series continued *in infinitum*: in the third case, when e is greater than 1, it will equally generate the series, though it will not express its arithmetical sum which, under such circumstances, is incapable of arithmetical expression: but it may be considered equally equivalent for some symbolical purposes at least, if not for all, whether it is arithmetically equal to it or not.

* Thus, if we should assume the existence of a series for $(1+x)^n$, proceeding according to positive powers of x, its first term would be necessarily 1, inasmuch as 1 is the value of $(1+x)^n$, when $x=0$: in a similar manner, the first term of the series for $(a+x)^n$, if proceeding according to positive powers of x, would be a^n: but the first term of the series, for the same expression, proceeding according to powers of $\frac{a-x}{a+x}$, would be $2a^n$, inasmuch as $2a^n$ would be the value of $(a+x)^n$, when $\frac{a-x}{a+x}=0$, or when $x=a$: in a similar manner, the first terms of series for $\cos x$, proceeding according to powers of x or of $\frac{\pi}{2}-x$, would be 1 in one case, and 0 in the other, inasmuch as $\cos x$ is 1 or 0, according as $x=0$ or $\frac{\pi}{2}$.

sign of the symbol or combination of symbols according to which it is arranged, we should at once conclude that its odd powers alone presented themselves in one series, and its even powers in the other: thus in assuming series, proceeding according to powers of x for $\sin x$ and $\cos x$, we should on this account, exclude all terms involving the even powers of x from one series, and all terms involving its odd powers from the other.

An equivalent series should possess as many values, if more than one, as the expression in which it originates.

959. A series, in order to be completely equivalent to the expression from which it is derived, should possess the same number of values with it, when those values are more than one: but the process of developement will in many cases apply exclusively, in the first instance, to the deduction of that form of the series which represents its arithmetical value, and which is *unique*, leaving the other values to be supplied by various methods, some of which will be explained hereafter: thus the series which the binomial theorem, as commonly applied, gives for $(1+x)^{\frac{1}{4}}$ is

$$1+\frac{x}{4}-\frac{3x^2}{4^2.1.2}+\frac{3.7x^3}{4^3.1.2.3}-\frac{3.7.11x^4}{4^4.1.2.3.4}+\ldots$$

which expresses the arithmetical value of the biquadratic root of $1+x$, when x is less than 1: its complete developement would be expressed by

$$1^{\frac{1}{4}}\left(1+\frac{x}{4}-\frac{3x^2}{4^2.1.2}+\frac{3.7x^3}{4^3.1.2.3}-\frac{3.7.11x^4}{4^4.1.2.3.4}+\ldots\right),$$

where $1^{\frac{1}{4}}$ is the *recipient* (Art. 724) of the multiple values of the biquadratic root, which may also be replaced by $\cos\frac{2r\pi}{4}+\sqrt{-1}$ $\sin\frac{2r\pi}{4}$, whose several values are 1, -1, $\sqrt{-1}$, and $-\sqrt{-1}$.

Similarly, we find

$$\sqrt[3]{(ax-x^2)}=(ax)^{\frac{1}{3}}\left\{1-\frac{x}{3a}-\frac{1.2.x^2}{3^2.1.2a^2}-\frac{1.2.5x^3}{3^3.1.2.3}-\ldots\right\},$$

where the value of the series, included between the brackets, is *unique*, but that of $(ax)^{\frac{1}{3}}$, which is multiplied into it, is *multiple:* the successive substitution of the three values, which $(ax)^{\frac{1}{3}}$ admits of, will give the triple series which represents the complete developement.

Whatever value a radical or

960. But if a radical or other term, admitting of multiple values, presents itself in a generating expression or *function*, and

subsequently reappears in the equivalent series into which it is developed, it should be kept in mind that, whatever be the value which it is assumed to possess on one side of the sign of equality, it must continue to retain it on the other: thus, in the series

other expression, admitting of multiple values, possesses in one term of a series, it retains in all.

$$\frac{1}{1+\sqrt{x}} = 1 - \sqrt{x} + x - x\sqrt{x} + x^2 - x^2\sqrt{x} + \ldots$$

whichever of its two values, $\sqrt{x}$, whether $+$ or -1, is assumed to possess in $\frac{1}{1+\sqrt{x}}$, the same is retained throughout the series: it is only, when such radical or other expressions (such as equisinal and equicosinal angles) present themselves in equations, whose members are neither identical nor reducible, by the performance of the operations indicated, to identity, and where no convention expressed or understood limits the multiplicity of their values, that we admit every combination of such values as being equally possible, and as being equally included within the range of the different cases to be considered.

The consideration of this subject, which is one of the most important in the theory of series, will be resumed in a subsequent Chapter.

961. When an expression, denoted by a symbol y, is developed in a series proceeding according to powers of any other symbol or quantity x, which it involves and upon which its value is dependent, we may *invert* the operation, and express x or the symbol according to which the first series is arranged, by means of another series, proceeding according to powers of the symbol y: thus if

Inversion of series.

$$y = A_1 x + A_2 x^2 + A_3 x^3 + \ldots$$

then the *inverse* series may assume the form

$$x = a_1 y + a_2 y^2 + a_3 y^3 + \ldots$$

and the problem proposed for solution is the following: "Given the first series or its successive coefficients A_1, A_2, $A_3 \ldots$ to find the second series or its successive coefficients a_1, a_2, $a_3 \ldots$." We shall exemplify it in the determination of the series for the measure of an angle in terms of its sine, the series for the sine of an angle in terms of its measure being given.

Inversion of the series for the sine of an angle in terms of its measure.

Since therefore (Art. 924)

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots \qquad (1),$$

we may assume, for the *inverse* series

$$x = A_0 \sin x + A_3 \sin^3 x + A_5 \sin^5 x + \dots \qquad (2):$$

for it is obvious that this series (2) must be confined to odd powers of $\sin x$, for the same reason that the series (1) for $\sin x$ was confined to odd powers of x: replacing in series (1), x and its powers by the series assumed to express it in (2), we get

$$\begin{aligned} x &= A_0 \sin x + A_3 \sin^3 x + A_5 \sin^5 x + \dots \\ -\frac{x^3}{1.2.3} &= \qquad\qquad -\frac{A_0^3 \sin^3 x}{1.2.3} - \frac{3A_0^2 A_3 \sin^5 x}{1.2.3} - \dots \\ +\frac{x^5}{1.2.3.4.5} &= \qquad\qquad\qquad\qquad + \frac{A_0^5 \sin^5 x}{1.2.3.4.5} + \dots \\ \dots\dots\dots &\quad \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Consequently, adding together the terms on both sides, we get

$$\sin x = A_0 \sin x + \left(A_3 - \frac{A_0^3}{1.2.3}\right) \sin^3 x + \left(A_5 - \frac{3A_0^2 A_3}{1.2.3} + \frac{A_0^5}{1.2.3.4.5}\right) \sin^5 x,$$

and equating the coefficients of corresponding terms, we find

$A_0 = 1$,

$A_3 - \dfrac{A_0^3}{1.2.3} = 0$, and therefore $A_3 = \dfrac{1}{1.2.3}$,

$A_5 - \dfrac{3A_0^2 A_3}{1.2.3} + \dfrac{A_0^5}{1.2.3.4.5} = 0$, and therefore $A_5 = \dfrac{9}{1.2.3.4.5}$,

....

and therefore

$$x = \sin x + \frac{\sin^3 x}{1.2.3} + \frac{9 \sin^5 x}{1.2.3.4.5} + \dots$$

The law of formation of the coefficients of this series is not made manifest by the number of its terms which have been thus determined, and it appears that the extension of the process to other terms would involve the formation of the successive powers of

$$A_0 \sin x + A_3 \sin^3 x + A_5 \sin^5 x + \dots$$

and requiring the aid of a theorem, called the *multinomial theorem,* which we have not hitherto investigated: in the progress of our investigations, however, we shall find other and more expeditious methods of effecting this and other developements, in which the laborious formation of complicated powers and products will be altogether avoided.

962. It has been shewn (Art. 778) that all measures of angles included in the expressions $2r\pi + x$ and $(2r+1)\pi - x$ are equisinal, and we may conclude that all such values of x are equally included in the equation

The inverse series for $\sin x$ is only true for the least of the measures of the equisinal angles.

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dots \qquad (1):$$

but it is the *least* of these equisinal values of x which is *alone* admissible in the *inverse* series

$$x = \sin x + \frac{\sin^3 x}{1 \cdot 2 \cdot 3} + \frac{9 \sin^5 x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

which is necessarily convergent and arithmetical in its value. This observation will be found hereafter to apply to *inverse* series and expressions generally, and is connected with important theories.

CHAPTER XLI.

ON THE SOLUTION AND THEORY OF CUBIC EQUATIONS.

The extraction of simple and compound cube roots is the solution of a cubic equation.

963. We have given, in a former Chapter, rules for the extraction of simple and compound cube roots (Art. 242), as far as those rules could be properly considered as included within the province of arithmetic: the processes in question are equivalent to the arithmetical solution of binomial and other cubic equations, in cases where they necessarily possess an arithmetical root, whose determination involves no ambiguity: we propose, in the present Chapter, to consider the general theory of the solution of cubic equations, and to exemplify the arithmetical or other rules to which it leads.

The second term of a complete cubic equation may always be obliterated.

964. A cubic equation, cleared of fractions and radical expressions, by the rules given in Chap. V., may be always reduced to the form

$$x^3 - ax^2 + bx - c = 0, \qquad (1)$$

where a, b, c are whole or fractional numbers, positive or negative: but this equation may be farther reduced by a very simple process, to a form in which its second term will disappear: for, if we make $x = y + \frac{a}{3}$, we get

$$\begin{aligned} x^3 &= y^3 + ay^2 + \frac{a^2y}{3} + \frac{a^3}{27}, \\ -ax^2 &= - ay^2 - \frac{2a^2y}{3} - \frac{a^3}{9}, \\ +bx &= + by + \frac{ab}{3}, \\ -c &= - c. \end{aligned}$$

Adding together the several terms on each side of the sign of equality, we find

$$x^3 - ax^2 + bx - c = y^3 - \left(\frac{a^2}{3} - b\right)y - \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right) = 0:$$

or, if we replace $\frac{a^2}{3} - b$ by q, and $\frac{2a^3}{27} - \frac{ab}{3} + c$ by r, we get

$$y^3 - qy - r = 0, \qquad (2)$$

$$\text{or } y^3 = qy + r^*. \qquad (3)$$

The signs of the roots of an equation may be changed.

965. If, in equation (3), we replace y by $-x$, we get

$$x^3 = qx - r; \qquad (4)$$

and it follows, therefore, that if a be an arithmetical root of equation (3), $-a$ is a symbolical root of equation (4), and conversely: the same process therefore which determines one of these roots will determine the other also, by passing from equation (3) to equation (4) or conversely, the negative roots of one equation being the positive roots of the other, and conversely: it is usual therefore, in the solution of cubic and other equations (as will be afterwards seen), to consider negative and positive roots as determined by the same arithmetical process, whenever such a process can be found: they are also called *real* roots, to distinguish them from those whose expression involves the sign $\sqrt{-1}$, and which are called *unreal* or *imaginary* roots.

Distinction of real and imaginary roots.

* Thus, the equation

$$y^3 + 6y^2 + 9y + 4 = 0$$

is reduced, by replacing y by $x - 2$, to

$$x^3 - 3x + 2 = 0.$$

The equation

$$x^3 + 21x^2 + 146x + 335 = 0$$

is reduced, by replacing x by $y - 7$, to

$$y^3 - y - 1 = 0.$$

The equation

$$y^3 + \frac{3y^2}{4} + \frac{11y}{16} + \frac{91}{192} = 0$$

is reduced, by replacing y by $x - \frac{1}{4}$, to

$$x^3 + \frac{x}{2} + \frac{1}{3} = 0.$$

In what sense the roots of equations may be classified as real and imaginary.

966. All roots, however, which admit of interpretation are equally real and significant, whether positive, negative, or affected with a sign involving $\sqrt{-1}$: but it will be useful, in the theory of equations, to recognize the preceding distinction of *real* and *imaginary* roots, as characterizing two great classes of roots, which are distinct from each other, both in the form in which they present themselves in equations, and in the practical arithmetical processes which are employed for their determination.

Solution of binomial cubic equations.

967. If we make ρ the arithmetical cube root of a (where a is a positive number, whole or fractional), there are, as we have seen (Art. 669), three symbolical cube roots of a, which are

$$\rho, \quad \left(\frac{-1+\sqrt{3}\sqrt{-1}}{2}\right)\rho, \quad \left(\frac{-1-\sqrt{3}\sqrt{-1}}{2}\right)\rho,$$

which are the three roots of the binomial cubic equation

$$x^3 - a = 0. \qquad (1)$$

In a similar manner, it will be found (Art. 670) that

$$-\rho, \quad \left(\frac{1+\sqrt{3}\sqrt{-1}}{2}\right)\rho, \quad \left(\frac{1-\sqrt{3}\sqrt{-1}}{2}\right)\rho$$

are the three symbolical cube roots of $-a$, or the three roots of the binomial cubic equation

$$x^3 + a = 0. \qquad (2)$$

The root ρ in equation (1) and $-\rho$ in equation (2), are called *real* roots, in conformity with the conventional language adopted in the last Article: the other roots, which are affected with the sign $\sqrt{-1}$, are called *imaginary* roots.

General process for the solution of cubic equations wanting the second term.

968. When q was a negative and r a positive whole number, we were able to determine a value of x which satisfied the equation

$$x^3 = qx + r \qquad (1)$$

by a direct arithmetical process (Arts. 240 and 241): and it may be shewn generally, if we assume

$$x = \sqrt[3]{s} + \sqrt[3]{t},$$

or x to be equal to the sum of the cube roots of s and t, that such values, whether arithmetical or symbolical, may be assigned to $\sqrt[3]{s}$ and $\sqrt[3]{t}$, as will satisfy the same equation for all values of q and r: for we thus get

$$\begin{aligned} x^3 &= (\sqrt[3]{s} + \sqrt[3]{t})^3 \\ &= 3\sqrt[3]{st}\,(\sqrt[3]{s} + \sqrt[3]{t}) + s + t \\ &= 3\sqrt[3]{st}\,.\,x + s + t, \qquad (2); \end{aligned}$$

replacing $\sqrt[3]{s} + \sqrt[3]{t}$ by x: but if the values of x and of x^3 be the same in equations (1) and (2), then

$$3\sqrt[3]{st}\,.\,x + s + t,$$

and

$$q x + r$$

are identical expressions, and therefore

$$3\sqrt[3]{st} = q, \text{ and } s + t = r;$$

or

$$st = \frac{q^3}{27}, \text{ and } s + t = r.$$

It will follow therefore that s and t are the roots of the given quadratic equation

$$u^2 - ru + \frac{q^3}{27} = 0\,*,$$

the solution of which gives us

$$s = \frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)},$$

$$t = \frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}.$$

We thus get

$$\begin{aligned} x &= \sqrt[3]{s} + \sqrt[3]{t} \\ &= \left\{\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}^{\frac{1}{3}} + \left\{\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}^{\frac{1}{3}}. \end{aligned}$$

If the equation proposed for solution had been

$$x^3 - ax^2 + bx - c = 0,$$

we must replace q by $\frac{a^2}{3} - b$, r by $\frac{2a^3}{27} - \frac{ab}{3} + c$, and add $\frac{a}{3}$, which gives

* For if s and t be the roots of the equation

$$u^2 - au + b = 0,$$

then $a = s + t$, and $b = st$. (Art. 661).

$$x = \frac{a}{3} + \left\{\frac{a^3}{27} - \frac{ab}{6} + \frac{c}{2} + \sqrt{\left(\frac{a^3 c}{27} + \frac{2a^2b^2}{27} - \frac{abc}{6} + \frac{b^3}{27} + \frac{c^2}{4}\right)}\right\}^{\frac{1}{3}}$$
$$+ \left\{\frac{a^3}{27} - \frac{ab}{6} + \frac{c}{3} - \sqrt{\left(\frac{a^3 c}{27} + \frac{2a^2b^2}{27} - \frac{abc}{6} + \frac{b^3}{27} + \frac{c^2}{4}\right)}\right\}^{\frac{1}{3}}.$$

This expression will shew how much the formula of the solution of a cubic equation is simplified, by the suppression of its second term.

The values of $\sqrt[3]{s}$ and $\sqrt[3]{t}$ are determined by the solution of an equation of six dimensions.

969. The values of s and t are derived, as we have seen, from the quadratic equation

$$u^2 - ru + \frac{q^3}{27} = 0,$$

and those of $\sqrt[3]{s}$ and of $\sqrt[3]{t}$ from the binomial cubic equations

$$v^3 - \frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = 0,$$

and

$$v^3 - \frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = 0:$$

or if we begin by representing $\sqrt[3]{s}$ by v and $\sqrt[3]{t}$ by z, we get

$$v^3 + z^3 = r \qquad (1),$$
$$3vz = q \qquad (2);$$

and if we replace z, in the first equation (1) by $\frac{q}{3v}$ derived from the second (2), we find

$$v^3 + \frac{q^3}{27v^3} = r,$$

$$\text{or } v^6 - rv^3 + \frac{q^3}{27} = 0,$$

an equation whose roots are the three values of

$$\left\{\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}^{\frac{1}{3}},$$

and the three values of

$$\left\{\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}^{\frac{1}{3}}:$$

it is in this sense that the solution of the equation

$$x^3 - qx - r = 0$$

may be said to be dependent upon that of an equation of 6 dimensions.

970. The general expression, however, for x, which is given in Art. 968 will be found to admit of more values than are compatible with the conditions which the sum of the cube roots of s and t are required to satisfy: for if we represent the arithmetical cube root of s by σ, and that of t by τ, and the three cube roots of 1 by 1, α, α^2 (Art. 669, Note,) then σ, $\alpha\sigma$, $\alpha^2\sigma$ are the three cube roots of s, and τ, $\alpha\tau$, $\alpha^2\tau$ are the three cube roots of t, and one of the first triplet of values may be combined with one of the second in nine different ways, as follows:

The general expression for the root admits of extraneous values which are not roots of the proposed equation.

$$\left\{\begin{array}{ll} 1. & \sigma+\tau. \\ 2. & \alpha\sigma+\alpha^2\tau. \\ 3. & \alpha^2\sigma+\alpha\tau. \end{array}\right.$$

$$\left\{\begin{array}{ll} 4. & \sigma+\alpha\tau. \\ 5. & \alpha\sigma+\tau. \\ 6. & \alpha^2\sigma+\alpha^2\tau. \end{array}\right.$$

$$\left\{\begin{array}{ll} 7. & \sigma+\alpha^2\tau. \\ 8. & \alpha^2\sigma+\tau. \\ 9. & \alpha\sigma+\alpha\tau. \end{array}\right.$$

But it should be observed that the equation of condition

$$3\sqrt[3]{s}\,\sqrt[3]{t}=q$$

restricts the selection of the combinations of such cube roots to those whose product is equal to $\frac{q}{3}$, and which are therefore independent of α or of those powers of α which are different from 1: it is obvious that this condition is satisfied by the *three first combinations only,* which exclusively represent therefore the three roots of the proposed cubic equation.

971. The three triplets of combinations of the cube roots of s and t, which are bracketted together above, will severally form the roots of the cubic equations

The nine combinations of the cube roots of s and t are the roots of an equation of nine dimensions.

$$x^3-qx-r=0,$$
$$x^3-\alpha qx-r=0,$$
$$x^3-\alpha^2qx-r=0:$$

and the nine values of those combinations will be the roots of the equation

$$(x^3-qx-r)(x^3-\alpha qx-r)(x^3-\alpha^2qx-r)=x^9-3rx^6+(3r^2-q^3)x^3-r^3=0.$$

It appears therefore that the expression which the preceding process has given for x, furnishes the general solution of an equation of 9 dimensions, of which the proposed cubic equation

$$x^3 - qx - r = 0$$

is a factor: and we are not authorized to conclude, from this investigation, that there is any symbolical expression which can be formed, which is capable of expressing simultaneously the three roots of the cubic equation

$$x^3 - qx - r = 0,$$

and those three roots only.

Cases in which there is only one real root which is arithmetical.

972. If r and q in the equation

$$x^3 - qx - r = 0$$

be both positive, and if $\frac{r^2}{4}$ be greater than $\frac{q^3}{27}$, then σ and τ (Art. 968) are arithmetical and real, and the three roots are

$$(1)\quad \sigma + \tau,$$

$$(2)\quad -\frac{\sigma + \tau}{2} + \frac{(\sigma - \tau)\sqrt{3}\sqrt{-1}}{2},$$

$$(3)\quad -\frac{\sigma + \tau}{2} - \frac{(\sigma - \tau)\sqrt{3}\sqrt{-1}}{2},$$

of which the first alone is arithmetical, the other two being imaginary.

Where one root only is real, but not arithmetical.

If r be negative, q positive, and $\frac{r^2}{4}$ greater than $\frac{q^3}{27}$, then σ and τ are negative and real: in this case there is only one real root, which is not arithmetical: the other roots are imaginary.

Where one root only is real which is arithmetical or not, according as r is positive or negative.

If q be negative, then σ and τ are real, but with different signs: there is, therefore, only one real root, which is arithmetical or not, according as r is positive or negative: the other roots are imaginary.

Case in which all the roots are real, and one only, or at most two, are arithmetical.

973. If q be positive, and $\frac{r^2}{4}$ less than $\frac{q^3}{27}$, then the two roots s and t of the reducing quadratic equation

$$u^2 - ru + \frac{q^3}{27} = 0$$

are imaginary: they have therefore no real and arithmetical cube roots.

If however, in this case, we make $c=\frac{r}{2}$ and $d=\sqrt{\left(\frac{q^3}{27}-\frac{r^2}{4}\right)}$, we shall find

$$x=(c+d\sqrt{-1})^{\frac{1}{3}}+(c-d\sqrt{-1})^{\frac{1}{3}}:$$

and if we make

$$\cos\theta=\cos(2r\pi\pm\theta)=\frac{c}{\sqrt{(c^2+d^2)}}=\frac{\frac{r}{2}}{\sqrt{\frac{q^3}{27}}},$$

we shall find (Art. 822)

$$x=2(c^2+d^2)^{\frac{1}{6}}\cos\frac{\theta}{3},\quad\text{or}\quad 2(c^2+d^2)^{\frac{1}{6}}\cos\frac{2\pi+\theta}{3},$$

$$\text{or}\quad 2(c^2+d^2)^{\frac{1}{6}}\cos\frac{2\pi-\theta}{3};$$

and, inasmuch as $(c^2+d^2)^{\frac{1}{6}}=\left(\frac{q^3}{27}\right)^{\frac{1}{6}}=\sqrt{\frac{q}{3}}$, the three values of x are

$$x'=2\sqrt{\frac{q}{3}}\cos\frac{\theta}{3},$$

$$x''=2\sqrt{\frac{q}{3}}\cos\frac{2\pi+\theta}{3},$$

$$x'''=2\sqrt{\frac{q}{3}}\cos\frac{2\pi-\theta}{3}.$$

These three roots of the equation are *real*, and one at least, but not more than two, of them, are arithmetical: for their sum is equal to zero*.

974. The term *irreducible* has been applied to the case of cubic equations considered in the last Article, where the three roots, though all of them are real and one of them, at least, arithmetical, are not capable of being determined, as in all other cases, by the extraction of roots, or the other processes of Arithmetic and Arithmetical Algebra: the difficulty which

The irreducible case of cubic equations: why so called.

* For $x'+x''+x'''=2\sqrt{\frac{q}{3}}\left\{\cos\frac{\theta}{3}+\cos\left(\frac{2\pi+\theta}{2}\right)+\cos\left(\frac{2\pi-\theta}{3}\right)\right\}$

$$=2\sqrt{\frac{q}{3}}\left(\cos\frac{\theta}{3}+2\cos\frac{2\pi}{3}\cos\frac{\theta}{3}\right)$$

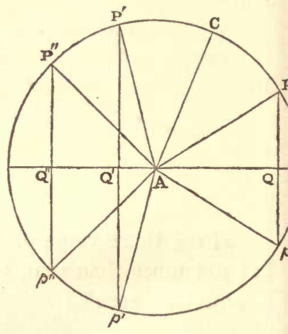
$$=2\sqrt{\frac{q}{3}}\left(\cos\frac{\theta}{3}-\cos\frac{\theta}{3}\right)=0.$$

was thus presented to the earlier writers on Algebra*, to whom the principles of Symbolical Algebra were almost entirely unknown, was insurmountable, and involved the solution, as the preceding investigation shews, of the problem of trisecting an angle, which was also beyond the province of plane Geometry: it is not the only case in which the separation of Arithmetical and Symbolical Algebra, and of plane and the higher Geometry in which the different conic sections and other curves appear, will be found to be marked by common limits.

Geometrical representation of the roots of a cubic equation in the irreducible case.

975. The principles of interpretation, which we have given in a former Chapter (XXXI.), will enable us to represent the two portions of which the several roots in the preceding solution are composed as well as the roots themselves.

Let the angle $BAC=\theta$, and let the angle $BAP=\frac{\theta}{3}=BAp$, the angle $BAP'=\frac{2\pi-\theta}{3}=BAp'$, and the angle $BAP''=\frac{2\pi+\theta}{3}=BAp''$: and join Pp, $P'p'$ and $P''p''$ cutting BA, and BA produced, in Q, Q', and Q'': if we assume the radius $AB=\sqrt{\frac{q}{3}}$, we shall find that



$$\sqrt{\frac{q}{3}}\left(\cos\frac{\theta}{3}+\sqrt{-1}\,\sin\frac{\theta}{3}\right) \text{ and } \sqrt{\frac{q}{3}}\left(\cos\frac{\theta}{3}-\sqrt{-1}\,\sin\frac{\theta}{3}\right),$$

will be expressed in magnitude and in position by AP and Ap, and their symbolical sum, or $2\sqrt{\frac{q}{3}}\cos\frac{\theta}{3}$ by $2AQ$: in a similar manner, the values of the other two roots will be the symbolical sums of AP' and Ap', and AP'' and Ap'', which are respectively equivalent to $2AQ'$ and $2AQ''$.

* An excellent account of the progress of the researches and discoveries of Tartalea, Cardan, Ferrari, Bombelli, Vieta, Des Cartes, and other early algebraists, in the solution of cubic and biquadratic equations, is given by Montucla in his *Histoire des Mathématiques*, Tom. I. Pars III. Livre III.

If the angle θ be between 0 and 90^0, or if its cosine be positive, then $\frac{2\pi+\theta}{3}$ and $\frac{2\pi-\theta}{3}$ are both greater than 90^0, and their cosines are negative: in other words, one root of the corresponding equation is positive, and two are negative: but if θ be between 90^0 and 180^0, or if its cosine be negative, then $\frac{\theta}{3}$ and $\frac{2\pi-\theta}{3}$ are less than 90^0, and $\frac{2\pi+\theta}{3}$ greater than 90^0; or in other words, two of the three real roots are positive, and one of them is negative. Case in which one or two of the three roots are arithmetical.

976. Before we proceed to the consideration of numerical examples of cubic equations, we shall notice the following problem as well calculated to illustrate the origin of their multiple solutions, and to exhibit the composition of their coefficients. Problem, illustrating the origin of the ambiguous solutions of cubic equations.

"To find three numbers, whose sum shall be equal to a, the sum of whose products shall be equal to b and their continued product equal to c."

Let x, y, z be the three numbers required; then the conditions of the problem give us

$$x+y+z=a,$$

$$xy+xz+yz=b,$$

$$xyz=c.$$

From the first equation we get

$$y+z=a-x,$$

and from the second

$$x(y+z)+yz=b,$$

$$\text{or } x(a-x)+yz=b,$$

$$\text{or } yz=b-ax+x^2:$$

the third equation gives us

$$x\times yz=x(b-ax+x^2)=c,$$

$$\text{or } x^3-ax^2+bx-c=0.$$

Inasmuch as x may represent *indifferently* any one of the three unknown numbers or quantities which were assumed to be represented by x, y, z, *which are all similarly involved in the original equations assumed,* it must equally represent them

all; or in other words, x will admit of three values, which are those of the several unknown symbols*.

If the sum of these three values or $a=0$, then one of them at least is positive, and two negative, or two of them positive, and one negative, when all the roots are real: or otherwise one of the three values is real, whether positive or negative, and the symbolical sum of the two others, which are imaginary, is equal to it with a different sign.

Composition of the coefficients of a cubic equation.

977. In considering the relations which the coefficients of a cubic equation

$$x^3 - ax^2 + bx - c = 0,$$

bear to its roots, it is evident likewise, from the result of the preceding elimination, (and the same may be easily proved from other considerations†), that the coefficient (a) of the second

* The same remark applies to the values of the unknown symbol in an equation, which results by elimination from any system of equations which are *symmetrical* with respect to the several unknown symbols which they involve: thus, if we eliminate y from the system of symmetrical equations

$$x^3 + y^3 = 35,$$
$$x + y = 5,$$

we get the quadratic equation

$$x^2 - 5x + 6 = 0,$$

where x has two values, which are those of x and y in the proposed system of equations; for x and y are obviously interchangeable with each other.

But if the system of equations had been

$$x^3 + y^3 = 35,$$
$$x - y = 1,$$

which are not symmetrical with respect to x and y, the equation resulting from the elimination of y would have been

$$x^3 - \frac{3}{2}x^2 + \frac{3x}{2} - 18 = 0,$$

a cubic equation, in which x has one real value only, the other two being imaginary.

† For, if α be a root of the equation

$$x^3 - ax^2 + bx - c = 0,$$

then $x - \alpha$ is a factor of it: for if

$$\alpha^3 - a\alpha^2 + b\alpha - c = 0,$$

then

$$x^3 - ax^2 + bx - c = x^3 - \alpha^3 - a(x^2 - \alpha^2) + b(x - \alpha),$$

which is divisible by $x - \alpha$:

term is the sum of the roots, the coefficient of the third term (b) is the sum of their products two and two, and the last term is their continued product: and it will therefore follow that if the second term disappear, the sum of the roots is necessarily equal to zero.

978. The following are examples of the solution of cubic equations.

(1) Let $x^3 - \frac{15x}{2} + \frac{581}{2} = 0$,

Since $\frac{r}{2} = \frac{581}{4}$, $\frac{q}{3} = \frac{5}{2}$, we get

$$\frac{r^2}{4} - \frac{q^3}{27} = \frac{337311}{16},$$

and therefore

$$s = -\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = -\frac{581 + \sqrt{337311}}{4},$$

$$t = -\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = -\frac{581 - \sqrt{337311}}{4},$$

$$\sigma = \sqrt[3]{\left(\frac{-581 + \sqrt{337311}}{4}\right)} = \frac{-7 + \sqrt{39}}{2},$$

$$\tau = \sqrt[3]{\left(\frac{-581 - \sqrt{337311}}{4}\right)} = \frac{-7 - \sqrt{39}}{2}.$$

Consequently

$$x' = \sigma + \tau = -7,$$

$$x'' = -\frac{(\sigma + \tau)}{2} + \frac{(\sigma - \tau)\sqrt{-3}}{2} = \frac{7}{2} + \frac{\sqrt{-117}}{2},$$

$$x''' = -\frac{(\sigma + \tau)}{2} - \frac{(\sigma - \tau)\sqrt{-3}}{2} = \frac{7}{2} - \frac{\sqrt{-117}}{2}.$$

In this example, we have put down the values of σ and τ, under a finite form, as determined by a *tentative* process of the following kind.

Tentative process for finding the cube roots of binomial surds, when such roots exist under a finite form.

and since α, β, and γ are roots of this equation, it thus appears that $x - \alpha$, $x - \beta$, and $x - \gamma$ are factors of it, and therefore there can be no more: we thus get

$$x^3 - ax^2 + bx - c = (x - \alpha)(x - \beta)(x - \gamma)$$
$$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma,$$

which proves the proposition in the text.

Thus, if it be suspected, from the nature of the case or otherwise, that a numerical expression of the form

$$a + \sqrt{b}$$

whose cube root is required, (as in the case of s and t in the Example just given) is a complete cube of an expression of the form

$$\frac{x + y}{\sqrt[6]{R}}$$

where x and y are, one or both of them, quadratic surds, and which are not reducible to a common quadratic surd, then we shall also have

$$\sqrt[3]{(a - \sqrt{b})} = \frac{x - y}{\sqrt[6]{R}}:$$

it will follow, therefore, that

$$\sqrt[3]{(a^2 - b)} = \frac{x^2 - y^2}{\sqrt[3]{R}},$$

$$\text{or} \quad \sqrt[3]{\{(a^2 - b)\, R\}} = x^2 - y^2 = c,$$

an expression, in which R may be always so assumed, that $(a^2 - b)\, R$ may be a perfect cube* and therefore $x^2 - y^2$ a rational number.

Again, since

$$\sqrt[3]{\{(a + \sqrt{b})^2\, R\}} = x^2 + 2xy + y^2,$$

$$\sqrt[3]{\{(a - \sqrt{b})^2\, R\}} = x^2 - 2xy + y^2,$$

we get

$$\sqrt[3]{\{(a + \sqrt{b})^2 R\}} + \sqrt[3]{\{(a - \sqrt{b})^2\, R\}} = 2x^2 + 2y^2:$$

and inasmuch as it would follow from the hypothesis which we have made, that x^2 and y^2 are whole numbers, the value of $2x^2 + 2y^2$ is necessarily equal to the sum of the two *nearest* integral values ι, ι' of $\sqrt[3]{\{(a + \sqrt{b})^2 R\}}$ and $\sqrt[3]{\{(a - \sqrt{b})^2 R\}}$, one of them being taken in defect and the other in excess: we thus find

$$x^2 + y^2 = \frac{\iota + \iota'}{2},$$

$$x^2 - y^2 = c.$$

* If $a^2 - b$ be not a perfect cube, or not resolvible into factors which are repeated, then the least value of R is $(a^2 - b)^2$: thus, if $u = 1162$, and $\sqrt{b} = 1349244$, we get

$$a^2 - b = 1000, \quad R = 1 \text{ and } x^2 - y^2 = 10:$$

$$\text{if } a^2 - b = 54 = 2 \times 27 = 2 \times 3^3, \text{ then } R = 4 \text{ and } x^2 - y^2 = 6:$$

$$\text{but if } a^2 - b = 58 = 2 \times 29, \text{ then we must make } R = 58^2 \text{ and } x^2 - y^2 = 58.$$

Adding and subtracting these expressions to and from each other, we get

$$2x^2 = \frac{\iota + \iota' + 2c}{2} \text{ and } x = \frac{\sqrt{(\iota + \iota' + 2c)}}{2},$$

$$2y^2 = \frac{\iota + \iota' - 2c}{2} \text{ and } y = \frac{\sqrt{(\iota + \iota' - 2c)}}{2}.$$

If, *upon trial*, the cube of $\frac{x+y}{\sqrt[6]{R}}$, which is thus determined, is found to be equal to $a + \sqrt{b}$, the problem is solved, and one of the roots of the equation is a whole number, or a rational fraction: if not, there is no such root of the equation, and its approximate value must be determined by the actual extraction of the roots involved in it.

It appears that unless $\sqrt{(\iota + \iota' + 2c)}$ and $\sqrt{(\iota + \iota' - 2c)}$ are even numbers, x^2 and y^2 cannot be whole numbers, but this may be avoided, by multiplying $a + \sqrt{b}$, in the first instance, by 2^3.

Thus, in the example under consideration, we find

$$s = -\frac{581 + \sqrt{(337311)}}{4} = -\frac{1162 + \sqrt{1349244}}{2^3},$$

$$t = -\frac{581 - \sqrt{(337311)}}{4} = -\frac{1162 - \sqrt{1349244}}{2^3},$$

and therefore $a = -1162$ and $b = \sqrt{1349244}$: consequently

$$a^2 - b = 1000 = 10^3:$$

we thus find $R = 1$, and

$$\sqrt[3]{(a^2 - b)} = x^2 - y^2 = 10.$$

Again $\quad a + \sqrt{b} = -1162 + 1161.569$

$$= -.431, \text{ nearly},$$

and therefore

$$\sqrt[3]{(a + \sqrt{b})^2} = .7554 \text{ nearly: say } 1:$$

and $\quad a - \sqrt{b} = -1162 - 1161.569,$

$$= -2323.569 \text{ nearly};$$

and therefore

$$\sqrt[3]{(a - \sqrt{b})^2} = 175.45 \text{ nearly: say } 175.$$

We thus get

$$x^2 - y^2 = 10,$$
$$x^2 + y^2 = 88,$$
$$x^2 = 49 \text{ and } x = -7,$$
$$y^2 = 39 \text{ and } y = \sqrt{39}.$$

Therefore

$$\sqrt[3]{(-1162+\sqrt{1349244})} = -7+\sqrt{39},$$

which upon trial is found to be true.

(2) Let $x^3 + 8x - 9 = 0$,

$$s = \frac{81+\sqrt{12705}}{18} = \frac{944784+\sqrt{1829520}}{6^3},$$

$$t = \frac{81-\sqrt{12705}}{18} = \frac{944784-\sqrt{1829520}}{6^3},$$

$$\sigma = \frac{3+\sqrt{105}}{6},$$

$$\tau = \frac{3-\sqrt{105}}{6};$$

and therefore

$$x' = 1, \quad x'' = \frac{-1+\sqrt{-35}}{2}, \quad x''' = \frac{-1-\sqrt{-35}}{2}.$$

(3) Let $x^3 - 9x + 14 = 0$,

$$s = -7+\sqrt{22} = -2.3096,$$

$$t = -7-\sqrt{22} = -11.6904,$$

$$\sigma = -\sqrt[3]{2.3096} = -1.3208,$$

$$\tau = -\sqrt[3]{11.6904} = -2.2696;$$

$$x' = \sigma + \tau = -3.5904,$$

$$x'' = -\frac{\sigma+\tau}{2} + \frac{(\sigma-\tau)}{2}\sqrt{-3} = 1.7952 + .82166\sqrt{-1},$$

$$x''' = -\frac{\sigma+\tau}{2} - \frac{(\sigma-\tau)}{2}\sqrt{-3} = 1.7952 - .82166\sqrt{-1}.$$

The process followed, in the two last Examples, for exhibiting the roots under a finite form, is not applicable in this case.

(4) Let $x^3 - 18x^2 + 87x - 70 = 0$.

If we make $x - 6 = u$ or $x = u + 6$, we get the transformed equation

$$u^3 - 21u + 20 = 0,$$

in which $\frac{r^2}{4} - \frac{q^3}{27} = 100 - 343 = -243.$

If we make $\cos\theta = \dfrac{-\frac{r}{2}}{\sqrt{\frac{q^3}{27}}} = \dfrac{-10}{7^{\frac{3}{2}}}$; and therefore

$$\cos(\pi - \theta) = \frac{10}{7^{\frac{3}{2}}};$$

we find

$$\begin{array}{lll} 10 + \log 10 & = 11, & \tfrac{1}{2}\log 7 = .4225490 \\ & & \phantom{\tfrac{1}{2}\log 7 = .42254}3 \\ \log 7^{\frac{3}{2}} & = 1.2676470 & \phantom{\tfrac{1}{2}\log 7 =\ }1.2676470 \\ \log\cos 57^{\circ}.19'.12'' & = 9.7323530 & \end{array}$$

and therefore

$$\theta = \pi - 57^{\circ}.19'.12'' = 122^{\circ}.40'.48''.$$

To find the value of $u' = 2\sqrt{\frac{q}{3}}\cos\frac{\theta}{3}$.

$$\begin{array}{ll} \log 2 & = \ .3010300 \\ \log\sqrt{\frac{q}{3}} = \frac{1}{2}\log 7 & = \ .4225490 \\ & \ \ .7235790 \\ \log\cos\frac{\theta}{3} = \log\cos 40^{\circ}.53'.36'' & = 9.8784812 \\ \log u' = \log 4 & = \ .6020600, \text{ omitting } 10. \end{array}$$

To find the value of $u'' = 2\sqrt{\frac{q}{3}}\cos\frac{2\pi - \theta}{3}$.

$$\begin{array}{ll} & \ \ .7235790 \\ \log\cos\frac{2\pi-\theta}{3} = \log\cos 79^{\circ}.6'.24'' & = 9.2764210 \\ \log u'' = \log 1 & = \ .0000000 \end{array}$$

To find the value of $u''' = 2\sqrt{\frac{q}{3}}\cos\frac{2\pi + \theta}{3}$.

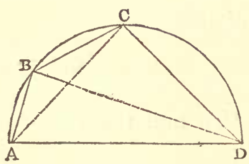
$$\begin{array}{ll} & .7235790 \\ -\log\cos\frac{2\pi+\theta}{3} = -\log\cos 160^\circ.53'.36'' = & 9.9753910 = \log\cos 19^\circ.6'.24'' \\ \hline -\log u''' = \log 5 \qquad\qquad = & .6989700 \end{array}$$

and therefore $u''' = -5$.

The corresponding values of x or the roots of the original equation are 10, 7, 1.

(5) Problem. Three consecutive chords, whose lengths are 1, 2, and 3 inches respectively, subtend a semicircle: to find the diameter of the circle.

Let the chords $AB = 1$, $BC = 2$, and $CD = 3$: let the diameter $AD = x$, and draw the diagonals AC (u), and DB (v): inasmuch as it appears, by a well-known property of quadrilateral figures, that



$$AC \times BD = AD \times BC + AB \times CD,$$

we get

$$uv = 2x + 3.$$

But, since $ABCD$ is a semicircle, the angles ABD and ACD are right angles, and, therefore,

$$u = \surd(x^2 - 1) \text{ and } v = \surd(x^2 - 9):$$

we thus get

$$\surd(x^2 - 1)\,\surd(x^2 - 9) = 2x + 3,$$

which gives, when rationalized

$$x^4 - 14x^2 - 12x = 0,$$

or dividing by x,

$$x^3 - 14x - 12 = 0^*.$$

* If the three chords had been expressed by a, b, c, the resulting equation would have been

$$x^3 - (a^2 + b^2 + c^2)\,x - abc = 0:$$

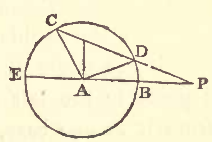
it is not difficult to prove that in this case $\frac{q^3}{27}$ is necessarily less than $\frac{r^2}{4}$. This problem is selected by Newton in his Arithmetica Universalis, Sectio 4ta. Cap. I., and solved in several different ways as an illustration of the various modes in which the same problem may be reduced to the form of an equation.

In this case $\frac{q^3}{27}$ is greater than $\frac{r^2}{4}$, and the process followed in the last example gives us

$$x' = 4.1133,$$
$$x'' = - .9118,$$
$$x''' = - 3.2015.$$

It is the positive root which expresses the *unique* value of the diameter, which solves the problem: the negative values are roots of solution only (Art. 676), and have no reference to the geometrical conditions of the problem proposed*: the solution is not therefore ambiguous.

(6) Problem. From the extremity of a given arc of a circle, to draw a line in such a manner that the portion of it, intercepted between the circle and a diameter produced which passes through the beginning of the arc, may be equal to the radius.



Let BC be the arc, and let CP be drawn to meet AB produced in P, in such a manner that $DP = AB$.

Make $AB = r$, $AP = x$, and the angle $BAC = \pi - \theta$: then since

$$PB \times PE = PD \times PC,$$

we get

$$PC = \frac{PB \times PE}{PD} = \frac{(x - r)(x + r)}{r} = \frac{x^2 - r^2}{r}:$$

but since

$$PC^2 = AP^2 + AC^2 - 2\,AP \times AC \cos(\pi - \theta),$$

we get

$$\frac{(x^2 - r^2)^2}{r^2} = x^2 + r^2 + 2\,r\,x \cos\theta,$$

which gives

$$x^4 - 3\,r^2 x^2 + 2\,r^3 \cos\theta\, x = 0,$$

$$\text{or } x^3 - 3\,r^2 x - 2\,r^3 \cos\theta = 0,$$

$$\text{or } \left(\frac{x}{r}\right)^3 - 3\left(\frac{x}{r}\right) - 2\cos\theta = 0.$$

* See Appendix.

Consequently

$$s = \cos\theta + \sqrt{-1}\,\sin\theta,$$

$$t = \cos\theta - \sqrt{-1}\,\sin\theta,$$

$$\frac{x'}{r} = 2\cos\frac{\theta}{3},$$

$$\frac{x''}{r} = 2\cos\frac{2\pi - \theta}{2},$$

$$\frac{x'''}{r} = 2\cos\frac{2\pi + \theta}{3}.$$

Thus, if $\theta = 60^0$, we find

$$\frac{x'}{r} = 2\cos\ 20^0 = \ \ .187949,$$

$$\frac{x''}{r} = 2\cos 100^0 = -\ \ .34740,$$

$$\frac{x'''}{r} = 2\cos 140^0 = -1.53209.$$

It is the arithmetical root 1.87949 which answers the conditions of the problem proposed: the other roots merely indicate that if the angle BAD be 100^0 in one case and 140^0 in the other, a point to the left of the centre A and at the distance $.34730r$ from it in one case, and $1.53209r$ in the other, may be found, whose distance from D is equal to r: and the angle BAD is $\frac{1}{3}^{\text{d}}$ EAC.

Difficulty of proving a universal negative in the geometrical solution of problems.

The geometrical problem, whose analytical solution we have just given, is one of the many forms to which the celebrated problem of the trisection of an angle has been reduced, but for which no construction, which can be effected by the rule and compass only, has been hitherto discovered: and it is presumed, from the failure of these attempts as well as from other considerations, that it is a problem whose solution is beyond the powers of plane geometry.

It is very difficult, however, if not impossible to shew, from geometrical considerations only, that the solution of this or of any other problem similarly circumstanced, is geometrically impossible, inasmuch as there is no limit to the number of constructions which may be proposed, all of which should be shewn to fail; and we are not enabled to conclude, by the forms of geometrical reasoning, from the failure of one construction, that

all others will be equally inadequate or inapplicable: it is for this reason that Geometers have not hitherto succeeded in assigning any form of demonstration, which is sufficiently conclusive and clear to check the attempts of geometrical adventurers, whose knowledge is imperfect or whose understandings are not sound and well disciplined, to solve impossible problems.

It will be found hereafter, however, that problems which admit of solution by straight lines or circles, or by combinations of them, will be, in all cases, reducible to simple or quadratic equations, or to such as are resolvible into them: and we shall be enabled to conclude therefore, that if the algebraical solution of a problem leads, as in the case just considered, to an *irreducible* cubic or higher equation, there exists no construction, which plane geometry can supply, which is capable of effecting it.

CHAPTER XLII.

ON THE THEORY AND SOLUTION OF BIQUADRATIC EQUATIONS.

General form of a biquadratic equation.

979. THE general form of a biquadratic equation is

$$x^4 - ax^3 + bx^2 - cx + d = 0 \qquad (1),$$

Its transformation so as to want its second term.

where a, b, c and d are rational numbers, whole or fractional, positive or negative; and it may, in all cases, be transformed, by the following process, into another biquadratic equation, whose roots differ from those of the given equation by a given number, but which shall want the second term.

For this purpose, we make $x - \frac{a}{4} = u$, or $x = u + \frac{a}{4}$: and therefore

$$\begin{array}{lllllll}
x^4 & = u^4 + au^3 & + \frac{3a^2u^2}{8} & + & \frac{a^3u}{16} & + & \frac{a^4}{256} \\[1ex]
-ax^3 & \quad - au^3 & - \frac{3a^2u^2}{4} & - & \frac{3a^3u}{16} & - & \frac{a^4}{64} \\[1ex]
+bx^2 & & + \; bu^2 & + & \frac{2bau}{4} & + & \frac{ba^2}{16} \\[1ex]
-cx & & & - & cu & - & \frac{ca}{4} \\[1ex]
+d & & & & & + & d.
\end{array}$$

If we add together the several terms on each side of the sign $=$, we get

$$0 = u^4 - \left(\frac{3a^2}{8} - b\right)u^2 - \left(\frac{a^3}{8} - \frac{ba}{2} + c\right)u - \left(\frac{3a^4}{256} - \frac{ba^2}{16} + \frac{ca}{4} - d\right);$$

and if we replace

$$\frac{3a^2}{8} - b \text{ by } q, \quad \frac{a^3}{8} - \frac{ba}{2} + c \text{ by } r,$$

and

$$\frac{3a^4}{256} - \frac{ba^2}{16} + \frac{ca}{4} - d \text{ by } s,$$

we get

$$u^4 - qu^2 - ru - s = 0. \qquad (2)^*$$

It is obvious that if the values of u can be determined by the solution of the transformed biquadratic equation, those of x in the original equation are immediately found, by adding $\frac{a}{4}$ severally to them: in the research, therefore, of general methods for the solution of biquadratic equations, we may confine our attention to those which want the second term, though it is not, in all cases, necessary to do so.

980. We have given, in a former Chapter (xx), many examples of the reduction of biquadratic equations, either wanting their alternate terms or possessing peculiar relations amongst their coefficients, to equations of an inferior degree: and we shall now proceed to apply one of the most common and successful of the expedients which are there exemplified, which consists in adding or subtracting such terms to or from both their members, as may make them complete squares.

Ferrari's method of resolving biquadratic equations.

* Thus the equation

$$x^4 - x^3 - 5x^2 + 12x - 6 = 0$$

becomes, by writing $y + \frac{1}{4}$ for x,

$$y^4 - \frac{43}{8}y^2 + \frac{75}{8}y - \frac{851}{256} = 0.$$

The equation

$$y^4 - 20y^3 + 148y^2 + 464x + 480 = 0$$

becomes, by putting $x + 5$ for y,

$$x^4 - 2x^2 + 16x - 15 = 0.$$

The equation

$$y^4 + 7y^3 - 19y - 20 = 0$$

becomes, by putting $x - \frac{7}{4}$ for y,

$$x^4 - \frac{147x^2}{8} + \frac{191x}{8} - \frac{3811}{256} = 0.$$

For this purpose, let the proposed biquadratic equation be put under the form

$$x^4 = qx^2 + rx + s; \qquad (1)$$

and inasmuch as the square of $x^2 + u$ is $x^4 + 2ux^2 + u^2$, let us add $2ux^2 + u^2$ to both its members: we thus get

$$(x^2 + u)^2 = (2u + q)x^2 + rx + u^2 + s \qquad (2).$$

In order that the second member of this equation may be also a complete square, it is necessary that we should have

$$4(2u + q)(u^2 + s) = r^2,\text{*}$$

or

$$2u^3 + qu^2 + 2su + qs = \frac{r^2}{4},$$

or

$$u^3 + \frac{q}{2}u^2 + su + \frac{qs}{2} - \frac{r^2}{8} = 0 \qquad (3).$$

It appears, therefore, that in order to determine the value of u, so that, if $2ux + u^2$ be added to both members of equation (1), they will severally become complete squares, it is necessary to solve the cubic equation

$$u^3 + \frac{q}{2}u^2 + su + \frac{qs}{2} - \frac{r^2}{8} = 0 \qquad (3).$$

If we suppose this equation solved, and a real value of u determined, we replace it in equation (2), and proceed as follows.

Extracting the square root on both sides of the equation, we get

$$x^2 + u = \pm(\sqrt{2u + q}\,.\,x + \sqrt{u^2 + s});$$

and we thus obtain the quadratic equations

$$x^2 - \sqrt{2u + q}\,.\,x + u - \sqrt{u^2 + s} = 0 \qquad (4),$$

or

$$x^2 + \sqrt{2u + q}\,.\,x + u + \sqrt{u^2 + s} = 0 \qquad (5),$$

according as we use the upper or lower sign: the solution of these two quadratic equations will give us the four values of x in the proposed biquadratic equation†.

* For if $(ax + b)^2 = a^2x^2 + 2abx + b^2$, then $4a^2 \times b^2 = (2ab)^2$.

† It was F. Ferrari of Milan, a disciple of Cardan's, who, about the middle of the 16th century, discovered this method of resolving biquadratic equations, on occasion of the proposition of the following problem.

Let the proposed equation be Examples.

$$x^4 - 6x^2 - 48x - 11 = 0.$$

The reducing cubic equation will be found to be

$$u^3 + 3u^2 + 11u - 255 = 0,$$

the real root of which, determined by the ordinary process, is 5.

The quadratic factors of the proposed equation are

$x^2 - 4x - 1 = 0$, whose roots are $2 + \sqrt{5}$ and $2 - \sqrt{5}$:

and

$x^2 + 4x + 11 = 0$, whose roots are $-2 + \sqrt{-7}$ and $-2 - \sqrt{-7}$.

Let the proposed equation be

$$x^4 - 25x^2 + 60x - 36 = 0.$$

The reducing cubic equation is

$$u^3 + 12 . 5u^2 + 36u = 0,$$

whose roots are 0, -8, and -4.5.

If we make $u = 0$, the system of quadratic equations is

$x^2 - 5x + 6 = 0$, whose roots are 2 and 3:

and

$x^2 + 5x - 6 = 0$, whose roots are 1 and -6.

For it appears that $25x^2 - 60x + 36$ is a complete square, and requires no addition to its terms to make it so.

If we make $u = -8$, the system of quadratic equations is

$x^2 - 3x + 2 = 0$, whose roots are 1 and 2:

and

$x^2 + 3x - 18 = 0$, whose roots are 3 and -6.

In this case, we add $-16x^2 + 64$ to $25x^2 - 60x + 36$, which produces the complete square $9x^2 - 60x + 100$.

If we make $u = -4.5$, the system of quadratic equations is

$x^2 - 4x + 3 = 0$, whose roots are 1 and 3:

"To find three numbers in continued proportion, whose sum is equal to 10, and the product of the first and second of which is equal to 6."

This problem leads immediately to the biquadratic equation

$$x^4 + 6x^2 - 60x + 36 = 0,$$

and to the reducing cubic equation

$$u^3 - 3u^2 - 36u - 342 = 0.$$

and

$$x^2 + 4x - 12 = 0, \text{ whose roots are } 2 \text{ and } -6.$$

In this case, we add $-9x^2 + 20.25$ to $25x^2 - 60x + 36$, which produces the complete square $16x^2 - 60x + 56.25$.

The roots of this biquadratic equation, as well as those of its reducing cubic, are all real, and the different solutions correspond to the three different ways of resolving it into quadratic factors, involving three different combinations of the roots; it is only when all the roots of the biquadratic equation are real that all those of the reducing cubic are real also.

The preceding process applicable to biquadratic equations, all whose terms are *complete*.

981. The process in the last Article is equally applicable to the solution of a biquadratic equation, all whose terms are complete: for if such a biquadratic equation be put under the form

$$x^4 - px^3 = qx^2 + rx + s; \qquad (1)$$

we shall find

$$\left(x^2 - \frac{px}{2} + u\right)^2 = x^4 - px^3 + \left(\frac{p^2}{4} + 2u\right)x^2 - pux + u^2,$$

which becomes, by replacing $x^4 - px^3$ by $qx^2 + rx + s$,

$$\left(x^2 - \frac{px}{2} + u\right)^2 = \left(\frac{p^2}{4} + q + 2u\right)x^2 - (pu - r)\,x + u^2 + s,$$

both sides of which are complete squares, if

$$4\left(\frac{p^2}{4} + q + 2u\right)(u^2 + s) = (pu - r)^2,$$

or if

$$u^3 + \frac{q}{2}u^2 + \left(s + \frac{pr}{4}\right)u + \frac{p^2 s - r^2}{8} + \frac{qs}{2} = 0 \qquad (2).$$

If the value of u, in equation (2) be found, the equation (1) is immediately resolvible, as in the former case, into two quadratic equations.

Example.

Thus, if it was required to find two numbers whose sum was 3, and the sum of whose fifth powers was 33, we should readily obtain the equation

$$x^4 - 6x^3 + 18x^2 - 27x + 14 = 0,$$

which would lead to the system of quadratic equations

$$x^2 - 3x + u = \pm(\sqrt{2u - 9} \cdot x + \sqrt{u^2 - 14}),$$

where u is found from the reducing cubic equation

$$u^3 - 9u^2 + \frac{53u}{2} - \frac{225}{8} = 0.$$

The real root of this equation is $\frac{9}{2}$, which gives x equal to

$$1,\ 2,\ \frac{3 \pm \sqrt{-19}}{2}.$$

The two numbers required are, therefore, 1 and 2.

982. The preceding method of solving biquadratic equations is altogether independent of any antecedent assumption concerning the existence of their roots, or the composition of their coefficients: if we conclude from it, however, as we are authorized to do, that every biquadratic equation

Composition of biquadratic equations.

$$x^4 - ax^3 + bx^2 - cx + d = 0$$

has four roots, and if we denote those roots by α, β, γ, d, then $x-\alpha$, $x-\beta$, $x-\gamma$, and $x-\delta$ may be easily shewn to be severally factors of the equation, and therefore

$$\begin{aligned}
&(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) - x^4 \\
&+ (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 \\
&+ (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma d \\
&= x^4 - ax^3 + bx^2 - cx + d:
\end{aligned}$$

we thus find, by equating corresponding terms of these identical results, that

$$a = \alpha+\beta+\gamma+\delta,$$

or the sum of the roots:

$$b = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta,$$

or the sum of all the products of the roots taken two and two:

$$c = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta,$$

or the sum of all the products of the roots taken three and three:

$$d = \alpha\beta\gamma\delta,$$

or the continued product of the roots.

It will follow therefore that if the coefficient of the second

term of a biquadratic equation is *zero*, the sum of the roots is zero*.

The same conclusion otherwise obtained.

983. The same conclusion would follow as in Art. 976 from the result of the elimination of three out of the four symbols x, y, z, v in the following system of symmetrical equations.

$$x + y + z + v = a \qquad (1),$$

$$xy + xz + xv + yz + yv + zv = b \qquad (2),$$

$$xyz + xyv + xzv + yzv = c \qquad (3),$$

$$xyzv = d \qquad (4).$$

From equation (1), we get

$$y + z + v = a - x.$$

From equation (2), we get

$$x\,(y + z + v) + yz + yv + zv = b,$$

and therefore

$$yz + yv + zv = b - x\,(a - x) = b - ax + x^2.$$

From equation (3), we get

$$x\,(yz + yv + zv) + yzv = c,$$

and therefore

$$yzv = c - x\,(b - ax + x^2) = c - bx + ax^2 - x^3.$$

And from equation (4), we also get

$$yzv = \frac{d}{x} = c - bx + cx^2 - x^3,$$

and therefore

$$x^4 - ax^3 + bx^2 - cx + d = 0.$$

The values of x in this equation are *quadruple*, for x may equally represent the value of x or y or z or v.

* The statement of this proposition supposes the terms of the original equation to be alternately positive and negative: if the form assumed had been

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

we should have found p equal the sum of the roots with their signs changed, and r equal to the sum of their products three and three with their signs changed.

If $\alpha, \beta, \gamma, \delta$ be the several values of x, y, z, and v, we then find as in Art. 982

$$\alpha + \beta + \gamma + \delta = a,$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = b,$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = c,$$

$$\alpha\beta\gamma\delta = d^*.$$

984. If we resume the consideration of the quadratic factors (4) and (5) in Art. 980, into which the equation Value of u in Art. 980, in terms of the roots of the biquadratic equation.

$$(x^2 + u)^2 - (2u + q)\,x^2 - rx - u^2 - s = 0$$

was resolved, and if we denote the pairs of roots of these factors, by α and β in one case, and by γ and δ in the other, we shall find

$$u + \sqrt{u^2 + s} = \alpha\beta,$$

$$u - \sqrt{u^2 + s} = \gamma\delta,$$

and therefore by addition

$$u = \frac{\alpha\beta + \gamma\delta}{2}:$$

and inasmuch, as we may interchange the roots $\alpha, \beta, \gamma, \delta$ in this expression at pleasure, the three values of u are

$$\frac{\alpha\beta + \gamma\delta}{2}, \quad \frac{\alpha\gamma + \beta\delta}{2}, \quad \frac{\alpha\delta + \beta\gamma}{2},$$

and *there are no more*†*:* we thus discover the reason why the values of u, upon which the determination of the roots of the biquadratic equation, in this method of solution, are dependent, are expressible by means of a cubic equation.

985. In the preceding process we have decomposed a biquadratic equation, under a modified form, into two quadratic Examination of the conditions

* The same method may be easily extended to explain the composition of the coefficients of an equation of n dimensions.

† For no more combinations can be formed by the interchange of the symbols $\alpha, \beta, \gamma, \delta$, in the semi-sums of these pairs of products.

which a quadratic factor of a biquadratic equation must satisfy.

factors*, by whose solution its roots are found: we shall now proceed to examine generally into the conditions which a trinomial such as

$$x^2 + ax + b \qquad (1),$$

must satisfy, in order that it may be a factor of the equation

$$x^4 - qx^2 - rx - s = 0 \qquad (2),$$

under its unaltered form.

For this purpose, we shall divide (2) by (1) as follows;

$$\begin{array}{l} x^2 + ax + b \big) \; x^4 - qx^2 - rx - s \; \big(x^2 - ax + a^2 - b - q \\ \qquad\qquad\quad x^4 + ax^3 + bx^2 \\ \hline \qquad\qquad\quad - ax^3 - (b+q)\,x^2 - rx \\ \qquad\qquad\quad - ax^3 - a^2x^2 - abx \\ \hline \qquad\qquad\qquad (a^2 - b - q)\,x^2 + (ab - r)\,x - s \\ \qquad\qquad\qquad (a^2 - b - q)\,x^2 + (a^3 - ab - aq)\,x + b\,(a^2 - b - q) \\ \hline \qquad\qquad\qquad\quad - (a^3 - 2ab - qa + r)\,x - b\,(a^2 - b - q) - s \end{array}$$

But if $x^2 + ax + b$ be one factor of $x^4 - qx^2 - rx - s$, then $x^2 - ax + a^2 - b - q$ must necessarily be the other†, and therefore the two terms of the remainder

$$-(a^3 - 2ab - aq + r)\,x - b\,(a^2 - b - q) - s$$

must be *identically* equal to *zero*: we thus get the two simultaneous equations

$$a^3 - 2ab - aq + r = 0 \qquad (3),$$

$$b\,(a^2 - b - q) + s = 0 \qquad (4).$$

From equation (3), we get

$$b = \tfrac{1}{2}(a^2 - q) + \frac{r}{2a},$$

* If $x^2 + ax + b$ is a factor of $x^4 - qx^2 - rx - s$, it continues to be a factor for all its values, and therefore when

$$x^2 + ax + b = 0,$$

in which case also

$$x^4 - qx^2 - rx - s = 0.$$

† For no negative power of x can appear in their product, nor, therefore, in the factors themselves.

and therefore (b')

$$a^2 - b - q = \tfrac{1}{2}(a^2 - q) - \frac{r}{2a}.$$

The substitution of these values in equation (4) gives us

$$\tfrac{1}{4}(a^2 - q)^2 - \frac{r^2}{4a^2} + s = 0,$$

which reduced, becomes

$$a^6 - 2qa^4 + (q^2 + 4s)\,a^2 - r^2 = 0 \qquad (5);$$

or, replacing a^2 by u

$$u^3 - 2qu^2 + (q^2 + 4s)\,u - r^2 = 0 \qquad (6),$$

a cubic equation, whose solution gives a value of u, and therefore of a^2.

A value of a being thus determined, the two quadratic factors of

$$x^4 - qx^2 - rx - s = 0$$

are

$$x^2 + ax + \tfrac{1}{2}\left(a^2 - q + \frac{r}{a}\right) = 0,$$

$$x^2 - ax + \tfrac{1}{2}\left(a^2 - q - \frac{r}{a}\right) = 0.$$

We thus obtain a method of determining the quadratic factors of a biquadratic equation, wanting its second term, without the necessity of any previous modification of its form. It was first given by the celebrated Des Cartes in his Geometry*, a work whose appearance formed a remarkable epoch in the history of Algebra and its applications.

Thus let the proposed equation be — Examples.

$$x^4 - 17x^2 - 20x - 6 = 0\,\dagger.$$

The reducing cubic equation is

$$u^3 - 34u^2 + 313u - 400 = 0,$$

one real root of which is 16, which gives $a = 4$: we thus get

$$b = \tfrac{1}{2}\left(a^2 - q - \frac{r}{a}\right) = -3,$$

$$b' = \tfrac{1}{2}\left(a^2 - q + \frac{r}{a}\right) = \;\; 2.$$

* Lib. III. † Ibid.

The component quadratic equations are therefore

$$x^2 + 4x + 2 = 0,$$

$$x^2 - 4x - 3 = 0,$$

whose roots are $-2 + \sqrt{2}$, $-2 - \sqrt{2}$, $2 + \sqrt{7}$ and $2 - \sqrt{7}$.

Euler's solution of a biquadratic equation and its deduction from that of Des Cartes.

986. If we denote the roots of the equation

$$x^4 - qx^2 - rx - s = 0$$

by $\alpha, \beta, \gamma, \delta$, we may replace δ by $-(\alpha + \beta + \gamma)$, inasmuch as their sum is equal to *zero*: but since a is the coefficient of the second term of one of the component quadratic equations, its value may be the sum of any two roots α, β, γ and $-(\alpha + \beta + \gamma)$ of the proposed biquadratic, and therefore equal either $\alpha + \beta$, $\alpha + \gamma$, $\beta + \gamma$, or, to $-(\alpha + \beta)$, $-(\alpha + \beta)$, $-(\beta + \gamma)$: the values of u or a^2 are therefore $(\alpha + \beta)^2$, $(\alpha + \gamma)^2$ and $(\beta + \gamma)^2$, which are only *three* in number and are, consequently, the roots of a cubic equation.

Again, if we suppose t, t', t'' to represent the three roots of the reducing cubic equation thus found or of

$$u^3 - 2qu^2 + (q^2 + 4s)u - r^2 = 0,$$

then, inasmuch as $t = (\alpha + \beta)^2$, $t' = (\alpha + \gamma)^2$, $t'' = (\beta + \gamma)^2$, we get

$$\sqrt{t} + \sqrt{t'} + \sqrt{t''} = (\alpha + \beta) + (\alpha + \gamma) + (\beta + \gamma)$$

$$= 2(\alpha + \beta + \gamma) = -2\delta,$$

$$\sqrt{t} - \sqrt{t'} - \sqrt{t''} = \alpha + \beta - (\alpha - \gamma) - (\beta + \gamma) = -2\gamma,$$

$$\sqrt{t'} - \sqrt{t} - \sqrt{t''} = \alpha + \gamma - (\alpha + \beta) - (\beta + \gamma) = -2\beta,$$

$$\sqrt{t''} - \sqrt{t} - \sqrt{t'} = \beta + \gamma - (\alpha + \beta) - (\alpha + \gamma) = -2\alpha.$$

It follows therefore that we can determine immediately the roots of the proposed biquadratic equation, by means of the roots of its reducing cubic equation

$$u^3 - 2qu^2 + (q^2 + 4s)u - r^2 = 0,$$

without the formation or solution of its component quadratic equations.

The same conclusion may be otherwise deduced by the following process, which was first given by Euler: let

$$x = \tfrac{1}{2}(\sqrt{t} + \sqrt{t'} + \sqrt{t''})$$

be assumed to represent a root of the equation

$$x^4 - qx^2 - rx - s = 0 \qquad (1),$$

and let t, t', t'' be the three roots of the cubic equation

$$u^3 - Pu^2 + Qu - R = 0 \qquad (2):$$

we thus get

$$x^2 = \tfrac{1}{4}(t + t' + t'' + 2\sqrt{tt'} + 2\sqrt{tt''} + 2\sqrt{t't''})$$
$$= \tfrac{1}{4}P + \tfrac{1}{2}(\sqrt{tt'} + \sqrt{tt''} + \sqrt{t't''})$$
$$\left(x^2 - \frac{P}{4}\right)^2 = \tfrac{1}{4}(tt' + tt'' + t't'' + 2\sqrt{tt't''}(\sqrt{t} + \sqrt{t'} + \sqrt{t''})$$
$$= \frac{Q}{4} + \sqrt{R}\,.\,x.$$

Transposing the significant terms to one side, we get

$$x^4 - \frac{P}{2}x^2 - \sqrt{R}\,.\,x + \frac{P^2 - 4Q}{16} = 0 \qquad (3).$$

Comparing the terms of the identical equations (1) and (3), we find

$$\frac{P}{2} = q, \quad \sqrt{R} = r, \quad \frac{P^2 - 4Q}{16} = -s,$$

$$\text{or } P = 2q, \quad R = r^2, \quad \frac{q^2 - Q}{4} = -s \text{ or } Q = q^2 + 4s,$$

and the reducing cubic equation (2), becomes

$$u^3 - 2qu^2 + (q^2 + 4s)u - r^2 = 0,$$

which is identical with the reducing cubic equation in Des Cartes's solution.

It should be observed that $\sqrt{tt't''} = \dfrac{R}{2}$ is an equation of condition, which makes it necessary that the values of $\sqrt{t}$, $\sqrt{t'}$, $\sqrt{t''}$ should be either all of them positive or two of them negative.

987. The following are examples of the application of this method of solution. Examples.

Let the equation proposed for solution be

$$x^4 - 25x^2 + 60x - 36 = 0.$$

The reducing cubic equation is

$$u^3 - 50u^2 + 769u - 3600 = 0.$$

The roots of this equation are

9, 16 and 25.

Therefore

$$\alpha = \frac{1}{2}(\sqrt{t} + \sqrt{t'} + \sqrt{t''}) = \frac{3}{2} \quad + 2 + \frac{5}{2} = \quad 6,$$

$$\beta \qquad = \frac{3}{2} \quad - 2 - \frac{5}{2} = -3,$$

$$\gamma \qquad = -\frac{3}{2} + 2 - \frac{5}{2} = -2,$$

$$\delta \qquad = -\frac{3}{2} - 2 + \frac{5}{2} = -1.$$

Again, let the proposed equation be

$$x^4 - 40x + 39 = 0.$$

The reducing cubic equation is

$$u^3 - 156u - 1600 = 0,$$

whose roots are 16, $-8 + 6\sqrt{-1}$, $-8 - 6\sqrt{-1}$.

Therefore

$$\alpha = \tfrac{1}{2}(4 + \sqrt{-8 + 6\sqrt{-1}} + \sqrt{-8 - 6\sqrt{-1}}),$$

$$= \tfrac{1}{2}\{4 + (1 + 3\sqrt{-1}) + (1 - 3\sqrt{-1})\} = 3,$$

$$\beta = \tfrac{1}{2}\{4 - (1 - 3\sqrt{-1}) - (1 - 3\sqrt{-1})\} = 1,$$

$$\gamma = \tfrac{1}{2}\{-4 + (1 + 3\sqrt{-1}) - (1 - 3\sqrt{-1})\} = -2 + 3\sqrt{-1}.$$

$$\delta = \tfrac{1}{2}\{-4 - (1 + 3\sqrt{-1}) + (1 - 3\sqrt{-1})\} = -2 - 3\sqrt{-1}.$$

A common principle involved in all methods of solving biquadratic equations.

988. The different methods of solving biquadratic equations, which we have given in the preceding Articles, have been found to depend upon the formation of a cubic equation, whose roots were determinate combinations of those of the equation required to be solved, whether with or without its second term: and inasmuch as there are only three roots of a cubic equation, those combinations alone would succeed, which admit of three values and no more: such combinations are

$$\frac{\alpha\beta + \gamma\delta}{2}, \quad \frac{\alpha\gamma + \beta\delta}{2}, \quad \frac{\alpha\delta + \beta\gamma}{2},$$

or

$$(\alpha + \beta)(\gamma + \delta), \quad (\alpha + \gamma)(\beta + \delta), \quad (\alpha + \delta)(\beta + \gamma),$$

or

$$(\alpha + \beta - \gamma - \delta)^2, \quad (\alpha + \gamma - \beta - \delta)^2, \quad (\alpha + \delta - \beta - \gamma)^2,$$

or

$$\frac{\alpha\beta - \gamma\delta}{\alpha + \beta - (\gamma + \delta)}, \quad \frac{\alpha\gamma - \beta\delta}{\alpha + \gamma - (\beta + \delta)}, \quad \frac{\alpha\delta - \beta\gamma}{\alpha + \delta - (\beta + \gamma)}\,^{*},$$

which are only three in number, whether α, β, γ and δ be the roots of a biquadratic equation, with or without its second term: such also are

$$(\alpha + \beta)^2, \quad (\alpha + \gamma)^2, \quad (\alpha + \delta)^2,$$

if the four roots are capable of being represented by α, β, γ and $-(\alpha + \beta + \gamma)$, as in the case of a biquadratic equation which wants its second term: we may conclude, therefore, as far as these examples will authorize us in doing so, that the methods of solving biquadratic equations (which will be considered under a more general point of view in Chap. XLV.) put us in possession of no specific process for the general solution of equations distinct from those of equations of the third degree†.

989. But it may be asked what is meant by the general solution of an equation of the third, fourth, or any higher order? *It is the discovery or determination of an expression, involving the coefficients of an equation when denoted by general symbols, whose multiple values shall express all the roots of that equation and no more: such equation being supposed to possess the most general form of its order, or such other form as is deducible from it by a general process.* It is a relaxation of the strictness of this definition of the meaning of the general solution of an equation, if we allow the multiple values of the expression or formula thus discovered, to be limited by means of one or more equations of condition: without such a restriction, the formula of Cardan is not a general solution of cubic equations, and the formula of solution of a quadratic equation would alone, of known methods, satisfy all the conditions required. What is meant by the general solution of equations.

When we speak of the numerical solution of an equation of any degree, we mean the discovery of a process which will enable us, in all cases, to determine its roots to any required degree of accuracy, when its general coefficients are replaced What is meant by the numerical solution of equations.

* These combinations form the basis of a very ingenious solution of a biquadratic equation which is given in the 1st volume of the Cambridge Mathematical Journal, a publication which is justly distinguished for the originality and elegance of its contributions to almost every department of analysis.

† See Waring, *Meditationes Algebraicæ*, p. 139.

by numbers; and it will be seen, in a subsequent volume of this work, how great is the progress which has been made in effecting the complete solution of this problem: but it should be always kept in mind that the general and the numerical solution of equations are problems of a different nature, and almost entirely independent of each other: in one case we seek to determine the roots one by one, by the repetition of a prescribed process: in the other, we are required to discover a single general formula, whose multiple values will express all the roots, and those roots only: and it should be observed that no such general formula has been found, even with the aid of equations of condition, for a general equation beyond the fourth degree.

But is the existence of such a formula, for equations of the fifth and higher degrees, possible; and if not, is there any simple mode of demonstrating that the conditions which it is required to satisfy are incompatible with the laws of Algebra? We have before remarked (Art. 978) upon the difficulty of proving a universally negative proposition, even in cases which involve relations of a very simple kind, and the attempts which have been made for this purpose by Abel and others are not calculated to lessen the impression which those observations are intended to convey: the clearest understanding gets bewildered by the extreme generality and complexity of the relations which it is necessary to consider in such investigations, and our final assent to the conclusion obtained is rather a formal act of acquiescence in reasonings whose entire force and relevancy we can neither fully appreciate nor easily refute than a spontaneous admission of a truth whose evidence is complete and irresistible.

It is contrary to probability that a formula should exist, formed by the fundamental operations of Algebra only, which should embrace all the coefficients of an equation, and express indifferently all its roots, and no more: for no such formula has hitherto been found, which rigorously fulfils those conditions beyond equations of the second degree: and the failure of the various attempts which have been made to discover it in the case of higher equations is more likely to be owing to the impossibility of the problem proposed than to the inadequacy of the methods which have been made use of for that purpose, or to want of skill in applying them.

CHAPTER XLIII.

ON THE SOLUTION AND REDUCTION OF RECURRING EQUATIONS.

Recurring and other equations which may be depressed to others of lower degree.

990. In the absence of general methods of solving equations of degrees superior to the fourth, there are many cases where equations present themselves with particular relations of their coefficients, which admit of reduction to binomial and other equations, whose degrees are within the limits of general solution: some of these we have considered in a former Chapter (xx): and another extensive class are denominated *recurring* equations, whose coefficients from the beginning and the end are the same; of this kind are the equations

$$x^5 + px^4 + qx^3 + qx^2 + px + 1 = 0,$$

and

$$x^6 + px^5 + qx^4 + rx^3 + qx^2 + px + 1 = 0,$$

the first of which is of *odd*, and the second of *even* dimensions.

A recurring equation of odd may always be depressed to one of even dimensions.

991. A *recurring* equation of *odd* dimensions may be always depressed to one of *even* dimensions of the next inferior degree, by dividing it by $x+1$ or $x-1$ according as its last term is 1 or -1: thus if we divide

$$x^5 + px^4 + qx^3 + qx^2 + px + 1 = 0$$

by $x+1$, we get

$$x^4 + (p-1)x^3 + (q-p+1)x^2 + (p-1)x + 1 = 0,$$

which is a recurring equation of 4 dimensions: in a similar manner, the recurring equation

$$x^7 - 10x^6 + 15x^5 - 20x^4 + 20x^3 - 15x^2 + 10x - 1 = 0$$

becomes, when divided by $x-1$,

$$x^6 - 9x^5 + 6x^4 - 14x^3 + 6x^3 - 9x + 1 = 0.$$

It appears, therefore, that in considering the theory of the solution of recurring equations, we may confine our attention to those which are of even dimensions only.

Depression of a recurring equation of 6 dimensions to a cubic equation.

992. If we take a recurring equation

$$x^6 + px^5 + qx^4 + rx^3 + qx^2 + px + 1 = 0$$

of 6 dimensions, we may reduce it to a non-recurring equation of half the number of dimensions: and a similar reduction may be effected in all similar cases: for if we divide the terms of this equation by x^3, and combine together the first and last term and also those which are equidistant from them, we shall get

$$x^3 + \frac{1}{x^3} + p\left(x^2 + \frac{1}{x^2}\right) + q\left(x + \frac{1}{x}\right) + r = 0:$$

and if we make

$$x + \frac{1}{x} = u,$$

we get

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = u^2 - 2,$$

and

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = u^3 - 3u:$$

the reduced equation thus becomes

$$u^3 - 3u + p(u^2 - 2) + qu + r = 0,$$

or

$$u^3 + pu^2 + (q - 3)u + r - 2p = 0,$$

which is a cubic equation.

Again, the recurring equation

$$x^8 + px^7 + qx^6 + rx^5 + sx^4 + rx^3 + qx^2 + px + 1 = 0$$

of eight dimensions, may be depressed, by a similar process, to the biquadratic equation

$$u^4 + pu^3 + (q - 4)u^2 + (r - 3p)u + s - 2q + 2 = 0.$$

The roots of recurring equations form pairs whose values are the reciprocals of each other.

993. It may further be observed that whatever be the value of u in the depressed equation, there are two values of x, corresponding to it, in the primitive equation, and which are the reciprocals of each other: for if $u = a$, we get

$$x + \frac{1}{x} = a,$$

and therefore

$$x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - 1\right)},$$

or

$$x = \frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - 1\right)} \text{ or } \frac{1}{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - 1\right)}}:$$

it follows therefore that if $a, b, c \ldots$ be roots of a recurring equation, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \ldots$ are also roots of it: such equations have, on this account, been sometimes called *reciprocal* equations.

994. There is another class of equations which admit of a similar reduction to others of half their dimensions, when those dimensions are even, which are sometimes called *quasi recurring* equations: of this class is the biquadratic equation Quasi recurring equations.

$$x^4 + px^3 + qx^2 + rx + \frac{r^2}{p^2} = 0,$$

where the last term is the quotient which arises from dividing the square of the coefficient of the last term but one by the square of the coefficient of the second term: for if we divide the several terms of this equation by x^2, and if we make

$$x + \frac{r}{px} = u,$$

we get the quadratic equation

$$u^2 + pu + q - \frac{2r}{p} = 0.$$

Of the same class is the equation

$$x^6 + px^5 + qx^4 + rx^3 + qsx^2 + ps^2x + s^3 = 0,$$

which may be depressed to a cubic equation by making

$$x + \frac{s}{x} = u.$$

995. A question may now arise, whether it is possible to transform the equation

$$x^4 - qx^2 - rx - s = 0$$

into a *quasi recurring* equation, such as that we are now considering: for this purpose, making $x = u + t$, we get the transformed equation (Art. 979)

$$u^4 + 4tu^3 + (6t^2 - q)u^2 + (4t^3 - 2qt - r)u + t^4 - qt^2 - rt - s = 0;$$

and in order that the required condition may be satisfied, we must assume t so that

$$\frac{(4t^3 - 2qt - r)^2}{16t^2} = t^4 - qt^2 - rt - s,$$

which leads to the cubic equation

$$t^3 + \frac{1}{2r}(q^2 + 4s)t^2 + \frac{qt}{2} + \frac{r}{8} = 0.$$

The real value of t determined from this equation, leads to the transformed equation

$$u^4 + Pu^3 + Qu^2 + Ru + \frac{R^2}{P^2} = 0,$$

which is depressible, by making $y = u + \dfrac{R}{Pu}$, to the quadratic equation

$$y^2 + Py + Q - \frac{2R}{P} = 0\text{*}.$$

Thus, if the proposed equation be

$$x^4 - 6x^2 - 48x - 11 = 0,$$

the reducing cubic equation is

$$t^3 + \frac{5t^2}{6} + 3t + 6 = 0,$$

one root of which is $-\dfrac{3}{2}$.

The transformed equation is therefore

$$u^4 - 6u^3 + \frac{15u^2}{2} - \frac{87u}{2} + \frac{841}{16} = 0.$$

If we make $y = u + \dfrac{R}{Pu} = u + \dfrac{87}{12u} = u + \dfrac{29}{4u}$, this equation becomes

$$y^2 - 6y - 7 = 0:$$

the values of y are 7 or -1, and those of x are $2 \pm \sqrt{5}$ and $-2 \pm \sqrt{-7}$.

Other equations whose roots are expressible by finite formulæ.

996. Other equations, which are not *quasi* recurring equations, but whose coefficients possess a particular relation to each other, may be solved by formulæ similar to those which occur in the general solution of a cubic equation. Thus if,

$$x^5 + \frac{s^5}{x^5} = a,$$

then

$$x^{10} - ax^5 = -s^5,$$

* Cambridge Mathematical Journal, Vol. I. p. 254.

and

$$x = \left\{\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}},$$

and therefore

$$x + \frac{s}{x} = \left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}} + \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}}.$$

But, if we make $u = x + \frac{s}{x}$, we get

$$u^5 = x^5 + \frac{s^5}{x^5} + 5s\left(x^3 + \frac{s^3}{x^3}\right) + 10s^2\left(x + \frac{1}{x}\right)$$

$$= a + 5s\,(u^3 - 3su) + 10s^2u,$$

which becomes

$$u^5 - 5su^3 + 5s^2u - a = 0,$$

an equation of the fifth degree whose roots have therefore, been determined.

If s and t be the arithmetical 5^{th} roots of $\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^5\right)}$, and $\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^5\right)}$ respectively, and if α be a root of $\frac{x^5 - 1}{x - 1} = 0$, then of the 25 values of u which are given by the preceding formula, s and t, αs and $\alpha^4 t$, $\alpha^4 s$ and αt, $\alpha^2 s$ and $\alpha^3 t$, and $\alpha^3 s$ and $\alpha^2 t$ are the only combinations which express the values of u in the proposed equation.

Again, the formula

$$\left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^6\right)}\right\}^{\frac{1}{6}} + \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^6\right)}\right\}^{\frac{1}{6}},$$

will express the roots of the equation

$$u^6 - 6su^4 + 9s^2u^2 - 2s^3 - a = 0:$$

and, in a similar manner, we may proceed to determine generally, the form of the equation whose roots are expressed by the formula

$$\left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^n\right)}\right\}^{\frac{1}{n}} + \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^n\right)}\right\}^{\frac{1}{n}},$$

both when n is an odd and an even number.

CHAPTER XLIV.

ON THE SOLUTION OF SIMULTANEOUS EQUATIONS, AND THE THEORY OF ELIMINATION.

Dependent and independent symbols.

997. The object proposed in the solution of those equations which have been considered in preceding Chapters of this work, has been to express one in terms of all the other symbols of the equation, upon whose values therefore, whether assigned or assignable, the value of this symbol is dependent, and by means of which it is determined: and the same necessary dependence between one symbol and all the others is presumed to exist in all equations whatsoever, whether the law and nature of such dependence is assignable or not: it is for this reason that such a symbol is termed the *unknown* quantity in the primitive equation, when all the other symbols are assigned or assignable, and when its determination is the object proposed by the solution of the equation; or it is termed the *dependent* symbol, when the others are not assigned, but are considered as perfectly arbitrary and independent of each other: in other words, the *dependent* symbol can only become known and determined, by assigning specific values of the several symbols, in terms of which it is expressed.

Connection of the character of dependent and independent symbols with the solution of the equation which involves them.

998. Thus the equation

$$ax + by - c = 0,$$

when solved with respect to x, gives us

$$x = \frac{c - by}{a},$$

where x is *dependent* upon y and the symbols a, b and c, which are all of them equally arbitrary and *independent*, so far as the conditions of the equation determine them: but if we suppose (as is most commonly the case), that the first letters of the alphabet a, b, c, denote quantities which are known or determinate, then x is the *dependent*, and y the *independent* symbol:

if, however, we should solve the primitive equation (1) with respect to y, we should get

$$y = \frac{c - ax}{b},$$

when y will become the *dependent*, and x the *independent* symbol: the character of *dependence* and *independence*, therefore, as distinguishing one symbol of an equation from the others, which are unknown and indeterminate, is convertible, and is determined by the solution, or *presumed* solution, of the equation with respect to one or other of those symbols.

999. The dependence of one symbol upon an independent symbol or symbols, whether accompanied by others which are known and determinate or not, is usually expressed by the term *function;* and the function (for the term is also used absolutely) is said to be *explicit* or *implicit,* according as the equation which involves the dependent and independent symbols, is solved with respect to the dependent symbol or not: thus x is an *explicit* function of y, in the equation

Meaning of the term function: explicit and implicit functions.

$$x = \frac{c - by}{a},$$

and y is an *explicit* function of x, in the equation

$$y = \frac{c - ax}{b};$$

but x is an *implicit* function of y, or y of x, (these relations being convertible), in the equation

$$ax + by - c = 0.$$

1000. The term *function* is not only used to denote the dependence of one symbol or quantity upon another or others, when their dependence is completely exhibited in a symbolical equation, whether *implicitly* or *explicitly,* but also when such dependence is *presumed* to exist, from the nature of the case, anteriorly to the investigation of any equation by which it may be expressed: thus if we should wish to express the dependence of the *space* described by a body when acted upon by *determinate* forces according to *determinate* laws, upon the *time* of its motion, we should say that the *space* was a *function* of the *time:* and if we should agree to denote the space by s, and

Presumed functions.

the time by t, we should express the same proposition by means of the equation $s = f(t)$, where the letter f, prefixed to the independent symbol (t), is used to express the term *function*.

The exhibition of the dependence of one symbol upon the others in an equation.

1001. When more than one unknown or indeterminate symbol presents itself in an equation, the solution of the equation with respect to any one of them, and therefore the exhibition of its *dependence* upon the others must be effected by the general methods which have been taught in the preceding Chapters, and must be limited by the limitation of those or other methods: thus the equation

$$x + y + \sqrt{(x+y)} = 12$$

may be reduced to the equivalent equation

$$x = 9 - y,$$

where the actual *dependence* of x upon y is exhibited: in a similar manner, the equation

$$\frac{x^2}{y^2} + \frac{2x+y}{\sqrt{y}} = 20 - \frac{y^2+x}{y}$$

leads to the equivalent equation

$$x = \frac{-y}{2} \pm \frac{9y}{2} - y^{\frac{3}{2}}:$$

and

$$x^4 - 2x^2y^2 + y^4 - 2a^2x^2 + 2a^2y^2 = b^4 - a^4,$$

to the equation

$$x = (a^2 \pm b^2 + y^2)^{\frac{1}{2}}.$$

Contributes nothing to the determination of their values.

It is obvious, however, that this exhibition of the dependence of one symbol upon one or more other indeterminate symbols leaves them necessarily as indeterminate as in the primitive equation: and however important such reductions may be for many of the purposes for which such equations may be applied, it can only be by the aid of other hypotheses or conditions, that their values can be absolutely determined*: such conditions, however various when considered in connection with the problems from which they arise, will generally resolve themselves into the *simultaneous* existence of as many inde-

* Such is the condition, considered in Chap. IX., which restricts the values of the unknown or indeterminate symbols to whole numbers.

pendent equations, as there are indeterminate, and in this case, unknown quantities involved in any one of them.

1002. Thus, if we have two equations involving x and y, possessing *simultaneous* values in both of them, then it is obvious that the value of x determined, in terms of y, from the first equation, must be the same as that of x determined, in terms of y, from the second: if we equate these values, we get an equation involving y only: the solution of this equation gives the absolute value or values of y, and leads us therefore necessarily to the absolute value or values of x: the following are examples, and many others have already been considered in Chap. v.

Process by which such equations may be solved or one or more unknown quantities eliminated.

Examples.

$$(1)\quad \left.\begin{aligned} 7x - 9y &= 7 \\ 3x + 10y &= 100 \end{aligned}\right\}.$$

The first equation, solved with respect to x, gives us

$$x = \frac{7+9y}{7}.$$

The second equation, solved likewise with respect to x, gives us

$$x = \frac{100-10y}{3}.$$

The values of x and y in the two equations being *assumed* to be identical, we must have

$$\frac{7+9y}{7} = \frac{100-10y}{3},$$

$$21 + 27y = 700 - 70y,$$

$$97y = 679,$$

$$y = 7,$$

and therefore $x = \dfrac{7+9y}{7} = \dfrac{7+63}{7} = \dfrac{70}{7} = 10.$

$$(2)\quad \left.\begin{aligned} \frac{x^2}{y^2} - \frac{4x}{y} + \frac{35}{9} &= 0 \\ x - y &= 2 \end{aligned}\right\}.$$

If we solve the first equation with respect to x, we get

$$x = \left\{2 + (1)^{\frac{1}{2}}\frac{1}{3}\right\} y.$$

If we solve the second equation with respect to x, we get

$$x = 2 + y.$$

If we equate these values of x, we get

$$\left\{2 + (1)^{\frac{1}{2}}\frac{1}{3}\right\} y = 2 + y:$$

$$\text{and therefore } y = 3 \text{ or } \frac{3}{2}.$$

The corresponding values of x are 5 and $\frac{7}{2}$.

$$(3) \quad \left.\begin{aligned} \frac{x+2}{5} - \frac{y+4}{8} + \frac{z+5}{10} - 1 = 0 \\ x + y + z - 12 = 0 \end{aligned}\right\}.$$

Clearing the first equation of fractions and solving it with respect to x, we get

$$x = \frac{5y - 4z + 24}{8}.$$

From the second equation, we get

$$x = 12 - y - z.$$

Equating these values of x, we get

$$\frac{5y - 4z + 24}{8} = 12 - y - z,$$

$$\text{or } 13y + 4z - 72 = 0.$$

We have thus reduced the two primitive equations to one, involving two unknown, and, in this case, *indeterminate* quantities: the third unknown quantity has been *eliminated* from them, and by the process employed for that purpose we have lessened the number of the proposed equations by 1.

If we had commenced by eliminating y, instead of x, from the primitive equations, we should have obtained the equation

$$13x + 9z - 84 = 0:$$

and if we had eliminated z, instead of x or y, we should have obtained the equation

$$4x - 9y + 24 = 0.$$

Elimination: final equation.

The *elimination* of symbols or unknown quantities, whether determinate or indeterminate, from equations in which they are involved, is one of the most important operations in Algebra,

and the processes which are employed for that purpose, which are very various in form, are generally identical with those employed for the solution of such equations, or rather for their reduction to a single final equation: if this final equation involves one unknown quantity only, its value may be determined absolutely, by means of it, by the aid of the known methods for solving equations, as far at least as those methods extend: if it involves two unknown quantities, they are both of them indeterminate, and one of them is independent and arbitrary: if the final equation involves more than two unknown and indeterminate symbols or quantities, then all but one of them are independent and arbitrary.

1003. In the preceding examples, the process of elimination of one unknown quantity has reduced the number of equations by one: and a very little consideration would shew that if n was the number of the primitive equations involving any number of unknown quantities, of which x was one, then the number of *independent* equations which would result from the elimination of x, would be $n-1$: for if each of these equations be solved with respect to x, (and we assume the practicability of such solutions) and if we should thus obtain

The number of any system of equations is diminished by the number of unknown quantities eliminated.

$$x = A_1, \quad x = A_2, \quad x = A_3, \ldots\ldots x = A_n,$$

where $A_1, A_2, A_3, \ldots\ldots A_n$ are the symbolical values of x derived from the several equations, then we should find, by equating the first value of x with each of the others, the following $(n-1)$ equations,

$$A_1 - A_2 = 0, \quad A_1 - A_3 = 0, \quad A_1 - A_4 = 0, \ldots\ldots A_1 - A_n = 0.$$

All other similar combinations of the quantities A_1, A_2, A_3, ... A_n with each other, though equally admissible with the preceding, will lead to equations immediately derivable from them, and therefore presenting no new and *independent* conditions for the determination of the unknown symbols which they involve: thus,

$$A_2 \;\;- A_3 = (A_1 - A_3) - (A_1 - A_2) \;\;= 0,$$

$$A_3 \;\;- A_5 = (A_1 - A_5) - (A_1 - A_3) \;\;= 0,$$

$$A_{n-1} - A_n = (A_1 - A_n) - (A_1 - A_{n-1}) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

in other words, the equation corresponding to any such combination will always arise from subtracting from each other some two of the $(n-1)$ equations which resulted from the first series of those combinations which it was considered most proper and convenient to select.

Again, inasmuch as the elimination of one unknown quantity from any number of (n) equations, diminishes the number of independent equations by 1, it will follow that the successive elimination of $(n-1)$ unknown quantities will diminish the number (n) of equations by $(n-1)$, and will therefore leave a single final equation remaining: if, therefore, the number of unknown quantities be the same as the number of equations which involve them, the final equation will involve one unknown quantity only, which will admit of determination by any methods which enable us to solve the equation itself: but if the number of unknown quantities exceeds the number of equations by m, the final equation will involve $(m+1)$ unknown quantities, which are therefore indeterminate in common with all the others, and m of them are *independent:* but if, on the contrary, the number of equations exceeds the number n of unknown quantities by m, all the unknown quantities may be determined from any assumed combination of n of those equations, which are

$$\frac{(n+m)(n+m-1)\ldots\ldots(m+1)}{1\cdot 2\ldots\ldots n}$$

in number: the values of the unknown quantities which would thus be obtained might be different for different combinations, and therefore m out of the $n+m$ equations are at least superfluous, if they are not inconsistent with each other. (Arts. 398, 402, 403).

Equations independent of each other.

When we speak of equations as independent of each other, we mean such as severally contain conditions for the determination of the unknown quantities which they involve, which are not supplied by the other equations, nor derivable from them: we must exclude therefore all equations which are multiples of any other of the equations proposed, when the multiplier is an assigned quantity, and independent of the unknown quantities involved in the equations: for the equation which thence arises can express no new condition for the determination of the quantities or symbols which are involved in it. Again, we must exclude equations which are multiples of other equations, when

the multiplier involves one or more unknown quantities, if such a multiplier be also a factor of another equation: for in such a case, the two equations will have a common measure (Art. 1005): lastly, we must exclude all such equations as are the sums or differences or products of the other equations, or of assigned multiples of them, inasmuch as all such equations are satisfied, by any values of the unknown quantities which satisfy the several equations which are involved in them, and consequently express no new and independent conditions for their determination.

In a system of equations it is not necessary that every equation should involve all the unknown quantities.

1004. It is not necessary that every equation should involve all the unknown quantities which are required to be determined: thus one equation may involve one unknown quantity, when its value is determined by means of it, or two unknown quantities or more: two equations may involve two unknown quantities, whose values will then be determined by means of them, or they may involve three unknown quantities or more: three equations may involve three unknown quantities, when their values are determined by means of them, or they may involve four unknown quantities or more: but the number of equations in any series of connected equations must not exceed, for reasons given above (Art. 1003), the number of unknown quantities involved in them, and no equation must involve more than one unknown quantity which does not also appear in other equations.

Suppression of common factors of equations.

1005. Any common factor, involving one or more unknown quantities, of two or more of a system of equations, may be detected by the methods employed for finding the highest common divisor, and must be excluded from them as foreign to the determination of the unknown quantities involved in them: for if such a factor involve one unknown quantity, any value of it which makes this factor zero, (and therefore determines it) will verify the equations which involve it, whatever be the values of the other unknown quantities; thus if there were two equations with two unknown quantities possessing such a factor, the other unknown quantity would remain perfectly indeterminate, and similarly in other cases. Again, if such a factor involve two or more unknown quantities, there are an infinite number of values which make this factor zero, and which therefore verify the equations which involve it, without any reference to the

values of the unknown quantities which the equation divided by this factor contain, and without contributing therefore in any respect to their determination; equations therefore which involve such common factors cannot be considered as independent of each other as long as such factors exist.

General process of elimination.

1006. It remains to consider the particular methods which are requisite for the *elimination* of unknown or indeterminate quantities from a system of equations: we shall confine our attention almost exclusively to a system of two equations with two unknown quantities, and in the first instance to the following method, which is one of the most general, though not always the most expeditious, of the various methods which have been proposed.

If there be two equations, $E_1 = 0$ and $E_2 = 0$, involving x and y, and possessing simultaneous values of them, then for every *proper* value of y, there must correspond a *proper* value of x: in other words, a *proper* value of y must be of such a kind, that if substituted in both the equations, the resulting equations involving x *only*, will have a *common* value or values of x: consequently, if E_1 and E_2 become X_1 and X_2 upon the substitution of a *proper* value of y, then X_1 and X_2 must have a common factor, which, when made equal to zero, will give the corresponding proper value or values of x: in order, therefore, to find such factors, we institute upon E_1 and E_2, arranged according to powers of x, the process for finding their highest common divisor, and continue it (excluding throughout fractional quotients and remainders) until we obtain a remainder Y, which involves y only: whatever value of y makes $Y=0$ will make the last divisor, which involves x, a common factor of E_1 and E_2, which, upon this substitution, become X_1 and X_2: if we find therefore all the values of y which make $Y=0$, we shall obtain all the corresponding values of x, and we shall thus be enabled to form all the sets or pairs of *proper* roots of the equations $E_1 = 0$ and $E_2 = 0$.

Circumstances which may modify the preceding process.

1007. The preceding is the statement of the general process which must be followed in such cases, without reference to circumstances which may sometimes modify the results which are obtained, or which may apparently cause us to fail in obtaining them: for in the first place, the process of finding the highest common divisor of E_1 and E_2 may introduce factors into Y, and

therefore values of y in $Y=0$, which are foreign to the system of equations: in the second place, two, three or more values of x may correspond to the same value of y, in which case the common factor of X_1 and X_2 will be of the form

$$x^2 + ax + b \text{ or } x^3 + ax^2 + bx + c,$$

and so on, and therefore the last divisor in the first case, the two last divisors in the second, and so on, will become equal to zero, for the corresponding proper value of y: these cases and their theory will be more particularly noticed amongst those examples, which follow, in which they first occur.

1008. Again, the system of equations may be themselves inconsistent with the existence of simultaneous values of x and y: this would be indicated by the final remainder or the last divisor becoming a numerical quantity which cannot be made equal to zero; circumstances which would indicate in one case that there was no value of x, which would give a common value of y, and in the second case, that there was no value of y, which would give a common value of x. Incompatible equations.

1009. Lastly, the final equation $Y=0$, may become an identical equation, in which case the last divisor, if not itself identically equal to zero, or the last divisor which is not so, must be a common factor of E_1 and E_2. This common factor must be excluded from the two primitive equations (Art. 1005) and the process instituted again with the quotients which result from its exclusion, which, when made equal severally to zero, become the independent equations which are the proper objects of consideration. Equations which have common factors involving x and y or both.

1010. The following examples will serve to illustrate the various propositions contained in the preceding articles. Examples.

(1) $ax + by - c = 0 = E_1$,

$\alpha x + \beta y - \gamma = 0 = E_2$,

$$
\begin{array}{l}
ax + by - c)\ \alpha x + \beta y - \gamma \\
\qquad\qquad\quad a \\
\hline
\qquad\qquad\quad a\alpha x + a\beta y - a\gamma\ (\alpha, \quad \text{(Art. 614)}, \\
\qquad\qquad\quad a\alpha x + \alpha b y - \alpha c \\
\hline
\qquad\qquad\quad (a\beta - \alpha b)\, y - (a\gamma - \alpha c) = Y.
\end{array}
$$

If $Y=0$, we find $y=\dfrac{a\gamma - \alpha c}{a\beta - \alpha b}$, and therefore $ax+by-c$ $=ax+b\left(\dfrac{a\gamma-\alpha c}{a\beta-\alpha b}\right)-c=0$, which gives us $x=\dfrac{\beta c - b\gamma}{a\beta-\alpha b}$.

If $a\beta - \alpha b = 0$, and if $a\gamma - \alpha c$ be *not* equal to zero, there is no common value of x in the two equations, which are therefore incompatible with each other.

If $a\beta - \alpha b = 0$, and also $a\gamma - \alpha c = 0$, then the two equations have a common measure which is independent of the particular values of x and y: under such circumstances the equations are *not* independent of each other, and the values of x and y continue indeterminate. (Art. 945).

(2) $\left.\begin{array}{l} x-y=5 \\ x^2+y^2=13 \end{array}\right\}$ or $\begin{array}{l} x+y-5=0=E_1 \\ x^2+y^2-13=0=E_2 \end{array}$

$$\begin{array}{rl}
x+y-5)\; x^2+y^2-13\;(x-y+5 & \\
x^2+(y-5)\,x & \\
\hline
-(y-5)\,x+y^2-13 & \\
-(y-5)\,x-y^2+10y-25 & \\
\hline
2y^2-10y+12=Y. &
\end{array}$$

If $Y=0$, or $2y^2-10y+12=0$, we find $y=3$ or 2: if $y=3$, we have $x+3-5=0$, and therefore $x=2$: if $y=2$, we have $x+2-5=0$, and therefore $x=3$.

(3) $x+y=a$ or $x+y-a=0=E_1$,

$x^3+y^3=b$ or $x^3+y^3-b=0=E_2$,

$$\begin{array}{rl}
x+y-a)\; x^3+y^3-b\;(x^2-(y-a)\,x+(y-a)^2 & \\
x^3+(y-a)\,x^2 & \\
\hline
-(y-a)\,x^2 & \\
-(y-a)\,x^2-(y-a)^2\,x & \\
\hline
(y-a)^2\,x+y^3-b & \\
(y-a)^2\,x+(y-a)^3 & \\
\hline
3ay^2-3a^2y+a^3-b=Y. &
\end{array}$$

If $Y=0$, we find

$$y=\frac{a}{2}+\sqrt{\left(\frac{4b-a^3}{12a}\right)}, \text{ and } \therefore x=\frac{a}{2}-\sqrt{\left(\frac{4b-a^3}{12a}\right)},$$

$$y=\frac{a}{2}-\sqrt{\left(\frac{4b-a^3}{12a}\right)}, \text{ and } \therefore x=\frac{a}{2}+\sqrt{\left(\frac{4b-a^3}{12a}\right)}.$$

The equation E_1 is of three dimensions, but the final equation $Y=0$ is of two dimensions only: it is obvious, however, that the second equation is reducible to one of two dimensions

$$x^2-xy+y^2-\frac{b}{a}=0,$$

by dividing x^3+y^3 by $x+y$ and b by a (Art. 976 and Note): the same remark is applicable to the dimensions of the final equation deducible from the two symmetrical equations $x+y=a$, and $x^5+y^5=b$, or to that from the two equations $x-y=0$, $x^4-y^4=0$, and similarly in other cases, where such a division of the equal members of one equation by the corresponding equal members of the other is found to be practicable.

(5) $\left.\begin{aligned} xy+xy^2-12=0=E_1 \\ x+xy^3-18=0=E_2 \end{aligned}\right\}$,

$$(1+y^3)\,x-18\}\,(y+y^2)\,x-12$$

$$\begin{aligned} &y\,(1+y^3)\,x-12\,(1-y+y^2)\,(y \\ &y\,(1+y^3)\,x-18y \\ \hline &-12y^2+30y-12=Y=0. \end{aligned}$$

Therefore

$$y^2-\frac{5y}{2}+1=0.$$

If $y=2$, then $x=2$.

If $y=\frac{1}{2}$, then $x=16$.

In this case, the coefficients of the first terms of the divisor and dividend have a common factor $1+y$, and therefore $y\,(1+y^3)$ is their *lowest* common multiple (Art. 619): if this circumstance was not attended to, the final equation would become

$$(y+1)\left(y^2-\frac{5y}{2}+1\right)=0,$$

and would involve a *foreign* factor $y+1$ and therefore a value of y to which no value of x would be found to correspond.

$$(6) \qquad \left.\begin{array}{r} yx^2 - 7x + 2 = 0 \\ (y-1)\,x^2 - 3x - 2 = 0 \end{array}\right\}.$$

$$yx^2 - 7x + 2\}\ (y-1)\,x^2 - 3x - 2$$

$$y\,(y-1)\,x^2 - 3yx - 2y\ (y-1$$

$$y\,(y-1)\,x^2 - 7\,(y-1)\,x + 2\,(y-1)$$

$$\overline{\qquad\qquad (4y-7)\,x - (4y-2)}$$

$$(4y-7)\,x - (4y-2))\ yx^2 - 7x + 2$$

$$(4y-7)^2\,yx^2 - 7\,(4y-7)^2\,x + 2\,(4y-7)^2$$

$$(4y-7)^2\,yx^2 - (4y-2)\,(4y-7)\,yx$$

$$\overline{(4y-7)(4y^2-30y+49)\,x + 2\,(4y-7)^2}$$

$$(4y-7)(4y^2-30y+49)\,x - (4y-2)(4y^2-30y+49)$$

$$\overline{\qquad\qquad 16y^3 - 96y^2 + 144y = 0 = Y.}$$

This final equation is reducible to the form $16y\,(y-3)^2 = 0$: if $y=3$, we have $x=2$: but if $y=0$ which is also a value of y in $Y=0$, we find no corresponding value of x: for under such circumstances E_1 and E_2 become $-7x+2$ and $-x^2-3x-2$, which have no common measure: but E_1 and yE_2 become, under the same circumstances, $-7x+2$ and 0, which are equally zero, when $y=0$ and $x=\frac{2}{7}$: it appears therefore that $y=0$ is a root of the final equaion, which is introduced by the process of solution, and which is altogether *foreign* to the equations $E_1=0$, and $E_2=0$, which were originally proposed.

Theory of foreign factors in the final equation.

If we consider this process generally, it would appear that $Y=0$, would necessarily contain the values of y which severally give common factors, not merely of E_1 and E_2 in their original form, but also of E_1 and E_2 one or both of them or of any of their successive remainders, when multiplied by such factors as are necessary to avoid the introduction of fractions: or, in other words, the final equation will be the same, as if such factors had existed as essential parts of the primitive equation or equations proposed: and if to the values of y which belong

to such factors, there are found corresponding values of x, then such values of y must necessarily present themselves as *foreign* roots of the final equation: but if there is no value of x corresponding to the values of y which belong to such factors, then the final equation cannot comprehend such values of y among its roots, and will be the same therefore as if such factors had never been introduced. It is for this reason, that the factors introduced for the last division, if the divisor involves the first power of x *only*, will never affect the degree of the final equation: for the values of x which belong to the factor whose introduction may be necessary in this case, will make the first term of the divisor equal to zero, and will leave therefore no term in which x exists: thus, in the example which we have been considering, if we make $(4y-7)^2=0$, the last divisor becomes -5, which is a numerical quantity.

No foreign factor introduced into the final equation, if the equations do not exceed the second degree.

For the same reason, likewise, it follows, that if the primitive equations do not exceed the second degree, no *foreign* root or factor can appear in the final equation: for if the first term of one or both of them involves x^2, it must present itself with a numerical coefficient only, and will not require therefore the introduction of any factor involving y: and it is obvious that the second divisor can involve x to the first degree only.

Suppression of factors in the primitive equation or the remainders.

It will frequently happen that a factor involving only y may be suppressed in the first or some subsequent remainder, and by such means the operation which is otherwise necessary, may be greatly shortened: if a value or values of x correspond to this factor, the simple factor itself, or powers of it, would appear, if not suppressed, in the final equation, according as the remainder in which it first appeared, was the remainder immediately preceding the final equation or not: for if it appeared in the last remainder but one, its square would be involved in the final equation; if in the last remainder but two, its cube, and so on; if therefore it is required to determine the *complete* final equation, such powers of this factor must be restored to it, at the conclusion of the operation: but if it is merely proposed, as is most commonly the case, to determine the different systems of values of x and y, it will be merely necessary to consider the value or values of y deduced from this factor in connection with those which are deduced from the final equation which results from its suppression: these observations would equally apply, if the factor in question was a factor of one of the primitive equations.

If in the example which has given occasion to these remarks, we had made $(y-1)x^2-3x-2$ the divisor, and yx^2-7x+2 the dividend, the final equation at which we should have arrived, would have been $(y-1)(y-3)^2=0$, where $y-1$, and not y, would be the foreign factor introduced by the operation.

$$(7) \quad x^2+(2y-7)x+y^2-7y-8=0,$$
$$x^2+(2y-5)x+y^2-5y-6=0.$$

If we subtract these equations from each other, we get

$$-2x-2y-2=0.$$

Therefore

$$x+y+1)\ x^2+(2y-5)x+y^2-5y-6\ (x$$
$$\underline{x^2+(\ y+1)x\qquad\qquad\qquad}$$
$$(y-6)x+y^2-5y-6$$
$$\underline{(y-6)x+y^2-5y-6}$$

Incompatible equations.

In this case there is no final equation, inasmuch as $x+y+1$ is a common factor, and therefore x and y are indeterminate; but if we take the equations

$$x+y-6=0,$$
$$x+y-8=0,$$

which result from the suppression of this factor, they are obviously incompatible with each other, or such as can be satisfied by no common values of x and y whatever.

In a similar manner it will be found that the equations

$$yx^3+y^2x^2+(y^2+y)yx+y^4+7y^2=0,$$
$$x^3-y^2+7x=0,$$

are incompatible with each other.

Process for the solution of three equations and three unknown quantities.

1011. When three equations, involving three unknown or indeterminate quantities are proposed for solution, we commence by reducing them to two equations with two unknown quantities, and subsequently to a single final equation, by a process similar to the one which has been employed in the preceding examples: thus let it be proposed to find the values of x, y, and z in the following equations:

$$\left.\begin{aligned} x - y + z - \quad 3 &= 0 \\ xy - z^2 + 10 &= 0 \\ x^2 + y^2 + z^2 - 29 &= 0 \end{aligned}\right\};$$

$$x - y + z - 3)\ yx - z^2 + 10$$

$$yx - y^2 + zy - 3y$$

$$y^2 - (z-3)\,y - z^2 + 10 = Y_1 = 0.$$

$$x - y + z - 3)\ x^2 + y^2 + z^2 - 29\ ($$

$$x^2 - yx + zx - 3x$$

$$(y - z + 3)\,x + y^2 + z^2 - 29$$

$$(y - z + 3)\,x - (y - z + 3)^2$$

$$2y^2 - 2(z-3)\,y + 2z^2 - 6z - 20 = 0$$

$$\text{or}\quad y^2 - (z-3)\,y + z^2 - 3z - 10 = Y_2 = 0.$$

If we now proceed to eliminate y from $Y_1 = 0$ and $Y_2 = 0$, we find, by subtracting one equation from the other,

$$y^2 - (z-3)\,y + z^2 - 3z - 10 = 0$$

$$y^2 - (z-3)\,y - z^2 + 10 = 0$$

$$2z^2 - 3z - 20 = Z = 0.$$

If we solve the final equation $Z = 0$, we get $z = 4$, or $-\frac{5}{2}$: the substitution of the first value of z in $Y_1 = 0$ or $Y_2 = 0$, gives $y = 3$ or -2: the substitution of the second value of z, gives $y = \frac{-11}{4} \pm \frac{\sqrt{61}}{4}$: the substitution of the corresponding pairs of the values of z and y in any one of the primitive equations, gives $x = 2$ or -3 or $\frac{11}{4} \pm \frac{\sqrt{61}}{4}$.

The values of x and $-y$ are symmetrically involved in the three equations, and the values of x and y, which are four in number, are therefore the same with different signs; if we had begun therefore by eliminating z from the three equations, we should have arrived at the following final biquadratic equations, according as the final equation was formed in x or in y.

$$x^4 - \frac{9x^3}{2} - \frac{31x^2}{4} + \frac{147}{4}x - \frac{45}{2} = 0,$$

$$y^4 + \frac{9y^3}{2} - \frac{31y^2}{4} - \frac{147}{4}y - \frac{45}{2} = 0.$$

Method of elimination by indeterminate multipliers.

1012. There are many other methods of elimination which peculiar circumstances will sometimes make more eligible than those which we have exemplified in the preceding Articles of this Chapter: of these, the method of Indeterminate Multipliers leads to processes which are not only symmetrical in their form, but also very easily applied.

For two equations.

Thus, if it was required to eliminate y from the equations

$$ax + by = c, \qquad (1)$$

$$a'x + b'y = c', \qquad (2)$$

we should multiply one of them by the *indeterminate multiplier* λ, which would give us the equations

$$ax + by = c, \qquad (1)$$

$$\lambda a'x + \lambda b'y = \lambda c'. \qquad (3)$$

If we add these equations together, we get

$$(a + \lambda a')x + (b + \lambda b')y = c + \lambda c': \qquad (4)$$

and if we make $b + \lambda b' = 0$ or $\lambda = -\frac{b'}{b}$, we get

$$\left(a - \frac{ba'}{b'}\right)x = c - \frac{bc'}{b'},$$

$$\text{or } (ab' - a'b)x = b'c - bc'.$$

For three equations.

1013. If there be three equations,

$$a_1x + b_1y + c_1z - k_1 = 0, \qquad (1)$$

$$a_2x + b_2y + c_2z - k_2 = 0, \qquad (2)$$

$$a_3x + b_3y + c_3z - k_3 = 0, \qquad (3)$$

we multiply the second by λ and add it to the first, which gives

$$(a_1 + a_2\lambda)x + (b_1 + b_2\lambda)y + (c_1 + c_2\lambda)z - (k_1 + k_2\lambda) = 0:$$

and if we make $c_1 + c_2\lambda = 0$, this equation becomes, when multiplied by c_2,

$$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y - (c_2k_1 - c_1k_2) = 0. \qquad (4)$$

In a similar manner, if we multiply the third equation by μ and add it to the first, we get, by making $c_1 + \mu c_3 = 0$ and multiplying by c_3,

$$(a_1 c_3 - a_3 c_1)x + (b_1 c_3 - b_3 c_1)y - (c_3 k_1 - c_1 k_3) = 0. \qquad (5)$$

If we now eliminate y from equations (4) and (5), as in the last Article, we get, after proper reductions,

$$x = \frac{k_1 b_2 c_3 - k_1 b_3 c_2 - k_2 b_1 c_3 + k_2 b_3 c_1 + k_3 b_1 c_2 - k_3 b_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1};$$

and, by a similar process, we likewise obtain

$$y = \frac{a_1 k_2 c_3 - a_1 k_3 c_2 - a_2 k_1 c_3 + a_2 k_3 c_1 + a_3 k_1 c_2 - a_3 k_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c^2 - a_3 b_2 c_1},$$

and

$$z = \frac{a_1 b_2 k_3 - a_1 b_3 k_2 - a_2 b_1 k_3 + a_2 b_3 k_1 + a_3 b_1 k_2 - a_3 b_2 k_1 {}^*}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1}:$$

To determine an equation of condition when the number of equations exceeds by 1 the number of quantities to be eliminated.

1014. If the number of equations $(n+1)$ exceed the number of unknown quantities (n) by 1, we may determine their values by means of n equations, and substituting those values in the $(n+1)^{\text{th}}$, we obtain an equation of condition, which must be satisfied, in order that the equations may be consistent with each other: thus, if we have the three equations

$$ax + by = c, \qquad (a)$$

$$a'x + b'y = c', \qquad (b)$$

$$a''x + b''y = c'': \qquad (c)$$

and only two unknown quantities x and y to be eliminated, we find from the two first

$$x = \frac{b'c - bc'}{ab' - a'b}, \quad y = \frac{a'c - ac'}{ab' - a'b},$$

which substituted in the third equation, give us, when multiplied by $ab' - a'b$, the equation of condition

$$a''(b'c - bc') + b''(a'c - ac') = c''(ab' - a'b). \qquad (d)$$

* See Art. 404. Appendix, Vol. I. p. 399.

The problem of elimination from equations of higher degrees reduced to elimination from equations of the first degree.

1015. If the coefficients, one or more of them, involve one or more unknown symbols, the equation of condition (d) will be the final equation to which the elimination of x and y, by this process, will lead: and it will be found to be very easy, by a very simple artifice, to extend this method of elimination to a system of equations in which different powers of one or more of the symbols which are required to be eliminated, present themselves*.

Thus, let the equations be

$$ax^2 + bx + c = 0, \qquad (1)$$

$$a'x^2 + bx' + c' = 0, \qquad (2)$$

where the coefficients a, b, c &c., one or more of them, involve y and its powers: and let it be required to eliminate x.

If we multiply these equations (1) and (2) by x, we get the additional equations

$$ax^3 + bx^2 + cx = 0, \qquad (3)$$

$$a'x^3 + b'x^2 + c'x = 0: \qquad (4)$$

and if we consider x^3, x^2, and x as distinct and independent symbols, like z, y, x, the number of equations, thus formed, will exceed by 1 the number of symbols to be eliminated.

If we eliminate x^3 from equations (3) and (4), we get the equation

$$(a'b - ab')x^2 + (a'c - ac')x = 0, \qquad (5)$$

$$\text{or } Ax^2 + Bx = 0,$$

replacing $a'b - ab'$ by A and $a'c - ac'$ by B.

If we now eliminate x^2 from equations (5) and (1) in one case, and from equations (5) and (2) in the other, we get

$$(Ab - aB)x + Ac = 0, \qquad (6)$$

$$(Ab' - a'B)x + Ac' = 0, \qquad (7)$$

which leads to the final equation or *determinant*

$$(Ab - aB)c' - (Ab' - a'B)c = 0:$$

or, replacing the values of A and B,

$$(ac' - a'c)^2 + (a'b - ab')(c'b - cb') = 0.$$

* *Cambridge Mathematical Journal*, Vol. II. p. 232 and 276.

Thus, if the proposed equations be

$$2x^2 - 3(y-1)x + y^2 - 3y - 6 = 0,$$

$$3x^2 - (3y-4)x - 10y^2 - 4 = 0,$$

the elimination of x, by this process, gives us the final equation

$$430y^4 - 252y^3 - 440y^2 + 150y + 112 = 0,$$

which is the same result as would be obtained by the method of the greatest common divisor.

It is this reduction of the general problem of elimination to that of elimination from equations of the first degree only, which gives additional importance to the latter problem, and to the research of the most easy and expeditious rules, both for the expression of the unknown symbols which they involve, or for the formation of the final equation, *under its most simple form,* which results from their elimination, when the equations exceed by one the number of such symbols: the discussion of the methods and formulæ to which this inquiry would lead, is one of no ordinary difficulty and extent, and is such as would hardly be consistent either with the object or the limits of this work: we shall confine ourselves therefore to the single problem of the result of the elimination of n unknown symbols from $n+1$ equations of the first degree.

1016. Resuming the expressions for x, y, z, which are given in Art. 1013, we shall be enabled to enunciate the law of formation of the numerators and denominators of those expressions, and to ascertain the principle upon which it may be extended so as to express the values of the unknown quantities in four or any greater number of such equations.

Enunciation of the law of formation of the expressions for the unknown symbols derived from three equations of the first degree.

In the first place, the numerator of each fraction differs from its denominator in having k, with its subscript numbers, in the place of the coefficient (with the same subscript numbers) of the unknown quantity whose value it expresses.

In the second place, the number of terms in each numerator and denominator is 6, or is equal to the number of permutations of the subscript numbers 1, 2, 3, (Art. 444).

In the third place, the algebraical sign of any term of the numerator or denominator will be the *same* with, or *different*

from, the first term in each, according as it involves an *odd* or an *even* number of quantities which are different from those in the first: thus, the terms $a_1 b_2 c_3$ and $a_1 b_3 c_2$, which involve two quantities, b_2, c_3, and b_3, c_2, which are different from each other, have *different* signs, whilst the terms $a_1 b_2 k_3$ and $a_2 b_3 k_1$, each involving three quantities which are different from each other, have the *same* sign.

It will follow as a consequence of the last observation, that the corresponding terms in the numerator and denominator, or, in other words, the terms with the same subscript numbers in the same order, will have the same algebraical signs.

Determination of the expressions for the unknown quantities in a system of four or of n equations.

We shall now proceed to consider the formation of corresponding expressions for the unknown quantities, when there are four or a greater number of such equations.

Let $x = \frac{N_1}{D}$, $y = \frac{N_2}{D}$, $z = \frac{N_3}{D}$, be assumed to represent the expressions for the three unknown quantities in three equations which are given above (Art. 1013); and let the four equations be

$$a_1 x + b_1 y + c_1 z + d_1 u - k_1 = 0 \;\ldots\ldots\; (1),$$

$$a_2 x + b_2 y + c_2 z + d_2 u - k_2 = 0 \;\ldots\ldots\; (2),$$

$$a_3 x + b_3 y + c_3 z + d_3 u - k_3 = 0 \;\ldots\ldots\; (3),$$

$$a_4 x + b_4 y + c_4 z + d_4 u - k_4 = 0 \;\ldots\ldots\; (4).$$

If in the expressions N_1, N_2, N_3 we replace k_1, k_2, k_3 by $k_1 - d_1 u$, $k_2 - d_2 u$, $k_3 - d_3 u$ respectively, and if n_1, n_2, n_3 be taken to represent the values of N_1, N_2, N_3, when k_1 is replaced by d_1, k_2 by d_2, k_3 by d_3, then we shall find, from the three first equations (1), (2), (3),

$$x = \frac{N_1 - n_1 u}{D}, \quad y = \frac{N_2 - n_2 u}{D}, \quad z = \frac{N_3 - n_3 u}{D};$$

and substituting these values in the last equation (4), and suppressing fractions, we get

$$(a_4 n_1 + b_4 n_2 + c_4 n_3 - d_4 D)\, u - a_4 N_1 - b_4 N_2 - c_4 N_3 + k_4 D = 0,$$

and therefore

$$u = \frac{a_4 N_1 + b_4 N_2 + c_4 N_3 - k_4 D}{a_4 n_1 + b_4 n_2 + c_4 n_3 - d_4 D}:$$

and if N', N'', N''' with their proper subscript numbers be successively taken to represent the values of the numerators of the expressions for any other three amongst the four unknown quantities in the three equations (1), (2), (3), namely, of

$x, y, u,$ when c is replaced by d and d by c,

$x, z, u,$ when b is replaced by c and c by b,

$y, z, u,$ when a is replaced by b and b by a;

and if the corresponding values of n and D be denoted by

$$n', \ n'', \ n''', \ D', \ D'', \ D''',$$

then we shall find

$$z = \frac{a_4 N_1' + b_4 N_2' + c_4 N_3' - k_4 D'}{a_4 n_1' + b_4 n_2' + c_4 n_3' - d_4 D'},$$

$$y = \frac{a_4 N_1'' + b_4 N_2'' + c_4 N_3'' - k_4 D''}{a_4 n_1'' + b_4 n_2'' + c_4 n_3'' - d_4 D''},$$

$$x = \frac{a_4 N_1''' + b_4 N_2''' + c_4 N_3''' - k_4 D'''}{a_4 n_1''' + b_4 n_2''' + c_4 n_3''' - d_4 D'''}.$$

The number of terms in the numerator and denominator of each of these expressions is four, or is equal to the number of unknown quantities: and if similar expressions were formed for five unknown quantities and five equations, they would severally contain five terms in their numerator and denominator; and similarly for whatever number of equations such expressions were investigated: for the substitution of the expressions $\frac{P_1}{Q}, \frac{P_2}{Q}, \ldots \frac{P_{n-1}}{Q}$ for $(n-1)$ unknown quantities derived from $(n-1)$ equations but *adapted* to a system of n equations and n unknown quantities, in the n^{th} or additional equation, will lead to an expression for the new unknown quantity, whose numerator and denominator consist of n terms; or if x_n denote the last unknown quantity introduced, and l and m (with their proper subscript numbers) be the coefficients of x_{n-1} and x_n, then we shall find

$$x_n = \frac{a_n P_1 + l_n P_2 + c_n P_3 + \ldots \, l_n P_{n-1} - k_n \, Q}{a_n Q + l_n Q + c_n Q + \ldots \, l_n Q \quad - m_n Q};$$

and it is obvious that corresponding symmetrical expressions

may be obtained for all the other unknown quantities x_{n-1}, x_{n-2}, ... x_1 in the inverse order of their introduction.

Enunciation of the general law of formation of the expressions for the n unknown quantities in a system of n equations.

It appears therefore that the number of factors in each product involved in the numerator and denominator of the expression for x_n will be n: for an additional factor is introduced for every additional unknown quantity or additional equation.

Again, the number of terms, when they are completely exhibited, in the numerator and denominator of the expression for x_n is $1 \times 2 \times 3 \times \ldots n$, or is equal to the number of permutations of the subscript numbers 1, 2, 3, ... n, (Art. 449): for the number of terms in the numerator and denominator of the expression for x_n is n times the number of them in the expression for x_{n-1} in a system of $(n-1)$ equations, $n(n-1)$ times the number of them in the expression for x_{n-2} in a system of $(n-2)$ equations, and so on, until we descend to the expression for the unknown quantity in a system of two equations.

Again, the number of positive and negative terms in the numerator and denominator being the same in the expression for the unknown quantity in a system of two and three equations, will continue the same likewise in the corresponding expressions in a system of n equations: for the new numerators of these expressions are formed by multiplying the series of literal numerators (which are the same as far as the letters involved and their signs are concerned) of the $(n-1)$ first unknown quantities and also their common denominator, into a_n, b_n, $c_n \ldots k_n$ respectively, and connecting their results with the sign +: if therefore the number of positive and negative terms be the same in the numerators and denominators of those expressions for $(n-1)$ unknown quantities and $(n-1)$ equations, it must continue the same therefore when there are n unknown quantities and n equations: and inasmuch as this number was the same, when there were two equations and two unknown quantities, it must continue the same therefore whatever be their number: the same observations apply, with a very trifling modification, to the number of positive and negative terms in the denominators.

Lastly, the same law which was noticed as determining the negative and positive terms in the case of the expressions for the unknown quantities in three equations, will prevail likewise for

any number of such equations: for whatever condition determines the sign of the separate terms in the numerators and denominators of the expressions for the unknown quantities, when there are $(n-1)$ unknown quantities, will determine their signs likewise when there are n unknown quantities: for the series of terms involved in their numerators and denominators are multiplied into new factors

$$a_n,\ b_n,\ c_n,\ \ldots\ldots\ k_n,$$

and therefore the conditions for the determination of the signs of the resulting products in each series remain the same as before.

1017. If it be required to find the final equation, when the number of equations $(n+1)$ exceeds by 1, the number of (n) symbols to be eliminated, we find from the n first equations,

Mode of obtaining the final equation when the number of symbols eliminated is less by 1 than the number of equations.

$$x = \frac{N}{D},$$

where N and D are formed by the general law which we have just investigated: we find, in a similar manner, from the n last equations,

$$x = \frac{N'}{D'},$$

where N' and D' are determined as before: we then equate the values of x, which are thus found, which gives

$$\frac{N}{D} = \frac{N'}{D'},$$

$$\text{or} \quad ND' - N'D = 0,$$

which is the final equation required.

Thus, if there be three equations,

$$ax + by - c = 0,$$

$$a'x + b'y - c' = 0,$$

$$a''x + b''y - c'' = 0,$$

we find from the two first equations,

$$(ab' - a'b)\,x - (b'c - bc') = 0,$$

and from the first and last,

$$(ab'' - a''b)\,x - (b''c - bc'') = 0,$$

which leads to the final equation

$$(ab' - a'b)\,(b''c - bc'') - (ab'' - a''b)\,(b'c - bc') = 0;$$

or, dividing by b,

$$ab'c'' - ab''c' - a'bc'' + a'b''c + a''bc' - a''bc' = 0.$$

It will be observed that this final equation is found by making the denominator $D = 0$, in the expression for any one of the three unknown symbols arising from the solution of three equations of the first degree which are given in Art. 1013.

CHAPTER XLV.

ON THE ALGEBRAICAL SOLUTION OF BINOMIAL EQUATIONS.

1018. We have shewn in a former Chapter, that the roots of the binomial equation

The goniometrical solution of the equation $x^n - 1 = 0$ does not imply its algebraical solution properly so called.

$$x^n - 1 = 0$$

are completely expressible by means of the different values of the formula

$$\cos \frac{2 r \pi}{u} + \sqrt{-1} \sin \frac{2 r \pi}{u},$$

where r may be replaced by any term of the series

$$0, 1, 2, 3, \ldots (r-1);$$

the same roots recurring, in the same order, if additional terms of this series are taken.

But this solution, though complete, is not algebraical, in the strict sense which is attached to that term, for reasons which have been explained before (Art. 990); but it will be found that the symbolical properties of those roots, and of the *cyclical* periods (Art. 722) which they form, will be sufficient to determine them algebraically or rather arithmetically in all cases.

For this purpose it will be necessary to investigate some propositions, which are equally applicable to the roots of binomial and other equations, and which will enable us to bring the various methods of solution of cubic and biquadratic equations, which we have already considered, under the operation of a common theory*.

1019. Let $\rho, \rho_1, \rho_2, \ldots \rho_{n-1}$ be the roots of any equation of n dimensions, and let $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ be the roots of the binomial equation

Formation of a general reducing equation from an assumed form of its root.

$$x^n - 1 = 0$$

* The substance of this theory is given by Lagrange, in notes XIII. and XIV. to the *Resolution des Equations Numeriques.*

of the same degree: and let it be required to investigate the conditions requisite for the determination of the roots of the equation which are expressed generally by the formula

$$t = \rho + \alpha\rho_1 + \alpha^2\rho_2 + \dots \alpha^{n-1}\rho_{n-1}\,*.$$

In the first place, the number of values of t, and therefore the dimensions of the corresponding reducing equation will be

$$1 \times 2 \times 3 \times \dots n:$$

for if we suppose

$$1,\ \alpha,\ \alpha^2 \dots \alpha^{n-1}$$

to retain their position, we may arrange the n roots

$$\rho,\ \rho_1,\ \rho_2 \dots \rho_{n-1}$$

in $1 \times 2 \times 3 \times \dots n$ different ways, each of which will produce a different value of t.

But a very little consideration will shew that these $1 \times 2 \times 3 \times \dots n$ roots or values of t may be distributed into

$$1 \times 2 \times 3 \times \dots (n-1)$$

groups of n roots each, whose n^{th} powers are identical: for if

$$t = \rho + \alpha\rho_1 + \alpha^2\rho_2 + \dots \alpha^{n-1}\rho_{n-1},$$

then also

$$\alpha t = \alpha\rho + \alpha^2\rho_1 + \alpha^3\rho_2 + \dots \rho_{n-1},$$

$$\alpha^2 t = \alpha^2\rho + \alpha^3\rho_1 + \alpha^4\rho_2 + \dots \alpha\rho_{n-1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\alpha^{n-1} t = \alpha^{n-1}\rho + \rho_1 + \alpha\rho_1 + \dots \alpha^{n-2}\rho_{n-1},$$

which are all of them equally included in the number of the $1 \times 2 \times 3 \times \dots n$ combinations which form the roots of the required equation: but inasmuch as

$$t^n = (\alpha t)^n = (\alpha^2 t)^n = \dots (\alpha^{n-1} t)^n,$$

it follows, that if we make

$$\theta = t^n$$

* This form of the root of the reducing equation, or *equation resolvante*, as he terms it, was suggested to Lagrange by an examination of the form of the roots of the reducing equations in the solutions of cubic and biquadratic equations: and it should be kept in mind that the propositions which follow have reference to the specific form assumed for t only or to such as are reducible to it, and therefore do not close the inquiry as far as other forms, which may be assumed for t, are concerned.

the number of different values of θ is only the $\frac{1}{n}^{\text{th}}$ part of the number of values of t, and consequently the values of θ are expressible by means of an equation of

The lowest dimension to which it is generally reducible.

$$1 \times 2 \times 3 \times \ldots (n-1)$$

dimensions.

We may assume, therefore,

$$\theta = t^n = A_0 + \alpha A_1 + \alpha^2 A_2 + \ldots \alpha^{n-1} A_{n-1},$$

where A_0, A_1, $A_2 \ldots A_{n-1}$, are combinations, whether *symmetrical** or not, of the roots ρ, ρ_1, $\rho_2 \ldots \rho_{n-1}$ of the equation to be solved: for it is obvious that θ can involve no power of α higher than α^{n-1}, inasmuch as all powers of α are reducible to some one term of the series

$$1, \alpha, \alpha^2 \ldots \alpha^{n-1}.$$

1020. If it be assumed *hypothetically*, that the values of A_0, A_1, $A_2 \ldots A_{n-1}$ are expressible by known numbers, or in terms of the coefficients of the equation to be solved, it may then be easily shewn that all its roots may be expressed in terms of the different values of $\sqrt[n]{\theta}$, which arise from replacing α by the successive roots of

The roots of the reducing will determine those of the proposed equation.

$$x^n - 1 = 0.$$

For if we denote the terms of the series

$$1, \alpha, \alpha^2 \ldots \alpha^{n-1}$$

by

$$1, r_1, r_2 \ldots r_{n-1},$$

and if we denote the corresponding values of θ in the equation

$$\theta = A_0 + \alpha A_1 + \alpha^2 A_2 + \ldots \alpha^{n-1} A_{n-1}$$

by

$$\theta_0, \theta_1, \theta_2 \ldots \theta_{n-1};$$

we shall find, since $\theta = t^n$, and therefore

* A *symmetrical* combination of the roots of an equation is *invariable*, inasmuch as it is not changed by a change in the order of the roots: and it may be easily proved that all symmetrical combinations of the roots of an equation are expressible in terms of its coefficients, and are therefore *given*, when the equation is *given*.

that

$$\sqrt[n]{\theta} = t = \rho + \alpha\rho_1 + \alpha^2\rho_2 + \ldots \alpha^{n-1}\rho_{n-1},$$

$$\sqrt[n]{\theta_0} = \rho + \rho_1 + \rho_2 + \cdots \rho_{n-1},$$

$$\sqrt[n]{\theta_1} = \rho + r_1\rho_1 + r_1^2\rho_2 + \ldots r_1^{n-1}\rho_{n-1},$$

$$\sqrt[n]{\theta_2} = \rho + r_2\rho_1 + r_2^2\rho_2 + \ldots r_2^{n-1}\rho_{n-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sqrt[n]{\theta_{n-1}} = \rho + r_{n-1}\rho_1 + r^2_{n-1}\rho_2 + \ldots r^{n-1}_{n-1}\rho_{n-1}.$$

If we add together the terms on both sides of the sign $=$, we get

$$\sqrt[n]{\theta_0} + \sqrt[n]{\theta_1} + \sqrt[n]{\theta_2} + \ldots \sqrt[n]{\theta_{n-1}} = n\rho^*,$$

or,

$$\rho = \frac{\sqrt[n]{\theta_0} + \sqrt[n]{\theta_1} + \sqrt[n]{\theta_2} + \ldots \sqrt[n]{\theta_{n-1}}}{n}.$$

Again, since

$$\sqrt[n]{\theta_0} = \rho + \rho_1 + \rho_2 + \ldots\ldots \rho_{n-1},$$

$$r_1^{n-1}\sqrt[n]{\theta_1} = r_1^{n-1}\rho + \rho_1 + r_1\rho_2 + \ldots r_1^{n-2}\rho_{n-1},$$

$$r_2^{n-1}\sqrt[n]{\theta_2} = r_2^{n-1}\rho + \rho_1 + r_2\rho_2 + \ldots r_2^{n-2}\rho_{n-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$r^{n-1}_{n-1}\sqrt[n]{\theta_{n-1}} = r^{n-1}_{n-1}\rho + \rho_1 + r_{n-1}\rho_2 + \ldots r^{n-2}_{n-1}\rho_{n-1},$$

we get

$$\sqrt[n]{\theta_0} + r_1^{n-1}\sqrt[n]{\theta_1} + r_2^{n-1}\sqrt[n]{\theta_2} + \ldots r^{n-1}_{n-1}\sqrt[n]{\theta_{n-1}} = n\rho_1,$$

or

$$\rho_1 = \frac{\sqrt[n]{\theta_0} + r_1^{n-1}\sqrt[n]{\theta_1} + r_2^{n-1}\sqrt[n]{\theta_2} + \ldots r^{n-1}_{n-1}\sqrt[n]{\theta_{n-1}}}{n}.$$

In a similar manner, we find

$$\rho_2 = \frac{\sqrt[n]{\theta_0} + r_1^{n-2}\sqrt[n]{\theta_1} + r_2^{n-2}\sqrt[n]{\theta_1} + \ldots r^{n-2}_{n-1}\sqrt[n]{\theta_{n-1}}}{n},$$

$$\rho_3 = \frac{\sqrt[n]{\theta_0} + r_1^{n-3}\sqrt[n]{\theta_1} + r_2^{n-3}\sqrt[n]{\theta_2} + \ldots r^{n-3}_{n-1}\sqrt[n]{\theta_{n-1}}}{n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\rho_{n-1} = \frac{\sqrt[n]{\theta_0} + r_1\sqrt[n]{\theta_1} + r_2\sqrt[n]{\theta_2} + \ldots r_{n-1}\sqrt[n]{\theta_{n-1}}}{n},$$

* For

$$1 + r_1 + r_2 + \ldots\ldots r_{n-1} = 0,$$

$$1 + r_1^2 + r_2^2 + \ldots\ldots r^2_{n-1} = 0,$$

$$1 + r_1^3 + r_2^3 + \ldots\ldots r^3_{n-1} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$1 + r_1^{n-1} + r_2^{n-1} + \ldots\ldots r^{n-1}_{n-1} = 0.$$

It thus appears that all the n roots of the proposed equation are successively expressible in terms of $\theta_0, \theta_1, \theta_2 \ldots \theta_{n-1}$ and of the n roots of

$$x^n - 1 = 0:$$

and it is in this sense, and in this sense only, that the general resolution of equations may be said to be dependent upon that of binomial equations of the same degree.

1021. Before we proceed with the developement of other parts of this theory, we shall proceed to apply it in the first instance to the solution of quadratic, and subsequently to those of cubic and biquadratic equations. Application of the preceding formulæ to the solution of a quadratic equation.

Let the quadratic equation be

$$x^2 - px + q = 0,$$

and let its roots be ρ and ρ_1: the roots of the corresponding binomial equation

$$x^2 - 1$$

are 1 and -1, and therefore $\alpha = -1$.

Let $t = \rho + \alpha\rho_1$,

$$\theta = t^2 = \rho^2 + \rho'^2 + 2\alpha\rho\rho' = A_0 + \alpha A_1.$$

But $\rho^2 + \rho'^2$ and $2\rho\rho'$, or A_0 and A_1 are symmetrical combinations of the roots, and are expressible therefore in terms of the coefficients of the proposed equation; we therefore find

$$A_0 = p^2 - 2q,$$

$$A_1 = 2q.$$

We thus get

$$\theta_0 = p^2 - 2q + 2q = p^2,$$

$$\theta_1 = p^2 - 2q + 2\alpha q = p^2 - 4q,$$

and therefore

$$\rho = \frac{\sqrt{\theta_0} + \sqrt{\theta_1}}{2} = \frac{p + \sqrt{(p^2 - 4q)}}{2},$$

$$\rho_1 = \frac{\sqrt{\theta_0} + \alpha\sqrt{\theta_1}}{2} = \frac{p - \sqrt{(p^2 - 4q)}}{2},$$

expressions which coincide with those given by the ordinary process of solution. (Art. 658.)

A modification of the form of the root of the reducing equation.

1022. It may be observed, that inasmuch, as when $\alpha = 1$, we have

$$t = \rho + \rho_1 + \rho_2 + \ldots \rho_{n-1},$$

or equal to the sum of the roots of the equation, which is always expressed by the coefficient of its second term, with its sign changed, it will follow therefore that, if p be this coefficient, we shall have

$$\theta_0 = (\rho + \rho_1 + \rho_2 + \ldots \rho_{n-1})^n = p^n,$$

which is therefore always known.

Again, inasmuch as

$$\theta_0 = A_0 + A_1 + A_2 + \ldots A_{n-1} = p^n,$$

we get

$$A_0 = p^n - A_1 - A_2 - \ldots A_{n-1}:$$

and if we substitute this value of A_0 in

$$\theta = A_0 + \alpha A_1 + \alpha^2 A_2 + \ldots \alpha^{n-1} A_{n-1},$$

it assumes the equivalent form

$$\theta = p^n + (\alpha - 1) A_1 + (\alpha^2 - 1) A_2 + \ldots (\alpha^{n-1} - 1) A_{n-1};$$

and therefore

$$\theta_1 = p^n + (r_1 - 1) A_1 + (r_1^2 - 1) A_2 + \ldots (r_1^{n-1} - 1) A_{n-1},$$

$$\theta_2 = p^n + (r_2 - 1) A_1 + (r_2^2 - 1) A_2 + \ldots (r_2^{n-1} - 1) A_{n-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\theta_{n-1} = p^n + (r_{n-1} - 1) A_1 + (r_{n-1}^2 - 1) A_2 + \ldots (r_{n-1}^{n-1} - 1) A_{n-1}.$$

Solution of a cubic equation.

1023. The great, and in most cases, insuperable difficulty of the problem consists in the determination of the values of $A_1, A_2 \ldots A_{n-1}$, or the coefficients of $\alpha, \alpha^2 \ldots \alpha^{n-1}$ in the expression for θ: thus in the cubic equation

$$x^3 - a x^2 + b x - c = 0,$$

if we make

$$t = \rho + \alpha\rho_1 + \alpha^2\rho_2,$$

we get

$$\theta = t^3 = A_0 + \alpha A_1 + \alpha^2 A_2,$$

$$= a^3 + (\alpha - 1) A_1 + (\alpha^2 - 1) A_2,$$

where

$$A_1 = 3(\rho^2\rho_1 + \rho_1^2\rho_2 + \rho_2^2\rho),$$

$$A_2 = 3(\rho^2\rho_2 + \rho_1^2\rho + \rho_2^2\rho_1),$$

which are convertible into each other, if we change ρ_1 into ρ_2, and ρ_2 into ρ_1: if we form the quadratic equation, therefore, of which A_1 and A_2 are the roots, we get

$$u^2 - Pu + Q = 0,$$

where P and Q are *invariable*, or in other words, involve the roots of the proposed equation *symmetrically*.

It is always possible, as will be shewn generally hereafter, to assign symmetrical expressions of the roots of an equation in terms of its coefficients, and the application of this theorem would enable us, in the case under consideration, to assign the values of P and Q in terms of the coefficients a, b, and c: but if with a view to the simplification of the very complicated expressions, which would thence result, we suppose $a = 0$, and if we replace b by $-q$ and c by r, we shall find

$$P = -9r,$$

$$Q = 9(9r^2 - q^3)^*,$$

and therefore

$$u = -27\left\{-\frac{r}{2} \pm \sqrt{3}\sqrt{\left(\frac{q^3}{27} - \frac{r^2}{4}\right)}\right\}.$$

We thus get

$$\theta_1 = (\alpha - 1)\, u_1 + (\alpha^2 - 1)\, u_2,$$

$$= 27\left\{-\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\},$$

$$\theta_2 = 27\left\{-\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\},$$

and therefore (Art. 1020)

$$\rho = \frac{\sqrt[3]{\theta_1} + \sqrt[3]{\theta_2}}{3},$$

$$\rho_1 = \frac{\alpha\sqrt[3]{\theta_1} + \alpha^2\sqrt[3]{\theta_2}}{3},$$

$$\rho_2 = \frac{\alpha^2\sqrt[3]{\theta} + \alpha\sqrt{\theta_2}}{3}.$$

These expressions coincide with those which are given in Art. 968.

* The complete expressions when the coefficients are p, q, r, are as follows:

$$P = -3pq - 9r,$$

$$Q = -9q^3 + 9(p^3 + 6pq)\, r + 81r^2.$$

An equation of n dimensions, when n is a prime number, is dependent, in its ultimate analysis, upon an equation of $1\times 2\times 3 \ldots (n-2)$ dimensions.

1024. More generally, if we consider the equation

$$x^n - px^{n-1} + p_1 x^{n-2} - \ldots \pm p_{n-1} = 0,$$

where n is a prime number, we find

$$\theta = p^n + (\alpha - 1) A_1 + (\alpha^2 - 1) A_2 + \ldots (\alpha^{n-1} - 1) A_{n-1},$$

which is given, as we have shewn in Art. 1019, by an equation of

$$1 \times 2 \times 3 \times \ldots (n-1)$$

dimensions: but if we form the equation

$$u^{n-1} - Pu^{n-2} + P_1 u^{n-3} - \ldots \mp P_{n-2} = 0,$$

whose roots are $A_1, A_2, \ldots A_{n-1}$, and if q be the number of different forms which this equation may assume for different values of $P, P_1 \ldots P_{n-2}$, then the final equation which arises from their continued product will have *invariable* coefficients, and will have $q(n-1)$ dimensions: and inasmuch as

$$q(n-1) = 1 \times 2 \times 3 \times \ldots (n-2)(n-1),$$

it follows that

$$q = 1 \times 2 \times 3 \times \ldots (n-2):$$

it appears, therefore, that whilst the values of θ are expressed by an equation of $1 \times 2 \times 3 \times \ldots (n-1)$ dimensions, the values $A_1, A_2 \ldots A_{n-1}$, in terms of which it is expressed, will be dependent upon an equation of $1 \times 2 \times 3 \times \ldots (n-2)$ dimensions.

In the case of cubic equations.

Thus, if $n=3$, as in the case of cubic equations, the values of θ are given by an equation of 1×2 dimensions, and the coefficients of the equation which expresses the values of A_1 and A_2 will depend upon an equation of the first degree, and they are therefore *unique:* this coincides with the result given in Art. 1023.

In the case of an equation of the 5th degree.

If n be 5, or if the proposed equation be of 5 dimensions, the values of θ will depend upon an equation of $1 \times 2 \times 3 \times 4$ or 24 dimensions, and the coefficients of the equation which expresses the values of A_1, A_2, A_3, A_4 will depend upon an equation of $1 \times 2 \times 3$ or 6 dimensions, exceeding by 1 degree the dimensions of the proposed equation.

When the number which expresses the

1025. If the number which expresses the dimensions of the proposed equation be not prime but composite, then it will be found that the process under consideration may be very greatly

simplified: for if $n = qm$, where m is not greater than q, then the series dimensions of the equation is not prime, but composite.

$$1,\ \alpha,\ \alpha^2 \ldots \alpha^{n-1}$$

of the roots of

$$x^n - 1 = 0$$

may be distributed into q groups or periods

$$1,\ \alpha,\ \alpha^2 \ldots \alpha^{m-1}, \quad 1,\ \alpha,\ \alpha^2 \ldots \alpha^{m-1} \ldots$$

recurring after every m^{th} term: we thus get

$$\begin{aligned} t = \rho + \alpha\rho_1 + & \ldots\ldots\ldots\ldots \alpha^{m-1}\rho_{m-1}, \\ + \rho_m + \alpha\rho_{m+1} + & \ldots\ldots\ldots \alpha^{m-1}\rho_{2m-1}, \\ + \rho_{2m} + \alpha\rho_{2m+1} + & \ldots\ldots\ldots \alpha^{m-1}\rho_{3m-1}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ + \rho_{(q-1)m} + \alpha\rho_{(q-1)m+1} + & \ldots\ \alpha^{m-1}\rho_{qm-1}: \end{aligned}$$

and if we make

$$\begin{aligned} \rho + \rho_m + \rho_{2m} + \ldots\ldots\ldots\ldots \rho_{(q-1)m} &= X_0, \\ \rho_1 + \rho_{m+1} + \rho_{2m+1} + \ldots\ldots \rho_{(q-1)m+1} &= X_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \rho_{m-1} + \rho_{2m-1} + \rho_{3m-1} + \ldots\ \rho_{qm-1} &= X_{m-1}, \end{aligned}$$

we shall find

$$t = X_0 + \alpha X_1 + \alpha^2 X_2 + \ldots\ \alpha^{m-1} X_{m-1}.$$

Considering therefore $X_0,\ X_1,\ X_2 \ldots X_{m-1}$ as the roots of a new equation of m dimensions only, we shall find

$$\begin{aligned} \theta = t^m &= A_0 + \alpha A_1 + \alpha^2 A_2 + \ldots\ldots\ldots\ \alpha^{m-1} A_{m-1}, \\ &= p^m + (\alpha - 1)\, A_1 + (\alpha^2 - 1)\, A_2 + \ldots\ (\alpha^{m-1} - 1)\, A_{m-1}, \end{aligned}$$

if p be the coefficient of the second term of the proposed equation with its sign changed: and therefore

$$X_0 = \frac{\sqrt[m]{\theta_0} + \sqrt[m]{\theta_1} + \sqrt[m]{\theta_2} + \ldots \sqrt[m]{\theta_{m-1}}}{m},$$

$$X_1 = \frac{\sqrt[m]{\theta_0} + r_1^{m-1}\sqrt[m]{\theta_1} + r_2^{m-1}\sqrt[m]{\theta_2} + \ldots r_{m-1}^{m-1}\sqrt[m]{\theta_{m-1}}}{m},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_{m-1} = \frac{\sqrt[m]{\theta_0} + r_1\sqrt[m]{\theta_1} + r_2\sqrt[m]{\theta_2} + \ldots\ r_{m-1}\sqrt[m]{\theta_{m-1}}}{m}.$$

Having thus determined the values of the groups of the roots of the proposed equation which are severally denoted by X_0, X_1, $X_2 \ldots X_{m-1}$, we proceed to apply the same process to the several equations of q dimensions which they form: and if q is still a composite number, we may again proceed to redistribute the roots of each primary into secondary groups, and to repeat the same process for their determination.

Dimensions of the final equation.

1026. The number (N) of values of t in the expression

$$t = X + \alpha X_1 + \alpha^2 X_2 + \ldots \alpha^{m-1} X_{m-1}$$

is

$$\frac{1 \times 2 \times 3 \times \ldots n}{(1 \times 2 \times 3 \times \ldots q)^m}:$$

for the number of permutations of the roots in each group is $1 \times 2 \times 3 \times \ldots q$, and therefore all the permutations which m such groups can form with each other is $(1 \times 2 \times 3 \times \ldots q)^m$: and if the number of values of t, when all the terms in the series

$$1, \alpha, \alpha^2 \ldots \alpha^{n-1}$$

are different from each other be $1 \times 2 \times 3 \times \ldots n$, and if the recurrence of the same period, after every m^{th} term, distributes them into groups admitting of $(1 \times 2 \times 3 \times \ldots q)^m$ permutations, which become identical with each other, it will follow that the number N of different values of t, under such circumstances, will be expressed by dividing $1 \times 2 \times 3 \times \ldots n$ by $(1 \times 2 \times 3 \times \ldots q)^m$.

Again, the number N' of different values of θ is only the $\frac{1}{m}^{\text{th}}$ part of the number of different values of t (Art. 1019), and therefore

$$N' = \frac{N}{m}.$$

Lastly, inasmuch as

$$\theta = p^m + (\alpha - 1) A_1 + (\alpha^2 - 1) A_2 + \ldots (\alpha^m - 1) A_{m-1},$$

it will follow that, if we form the equation

$$u^{m-1} - P u^{m-2} + Q u^{m-3} - \ldots = 0,$$

whose roots are A_1, $A_2 \ldots A_{m-1}$, the number N'' of sets of such coefficients (arising from different values of A_1, $A_2 \ldots A_{m-1}$)

which enter into them will be $(m-1)N''$, which is also equal to N': we thus get

$$N'' = \frac{N'}{m-1} = \frac{N}{m(m-1)}$$

$$= \frac{1 \times 2 \times 3 \times \ldots n}{m(m-1)(1 \times 2 \times 3 \times \ldots q)^m},$$

a number which expresses the dimensions of the final reducing equation upon which the formation of the coefficients P, Q, R, &c., and therefore the solution of the proposed equation, in its ultimate analysis, may be said to depend.

1027. Let us apply the formulæ in the last Article to the solution of the biquadratic equation

Application of the preceding theory to the solution of a biquadratic equation.

$$x^4 - px^3 + qx^2 - rx + s = 0 \qquad (1).$$

In this case we have $m = q = 2$, and α or the root of

$$x^2 - 1 = 0,$$

which is different from 1, is -1: we thus find

$$t = X_1 + \alpha X_2, \text{ where } X_1 = \rho + \rho_2 \text{ and } X_2 = \rho_1 + \rho_3:$$

and

$$\theta = t^2 = A_0 + \alpha A_1 = p^2 + (\alpha - 1) A_1.$$

The *triple* values of $A_1 = 2X_1X_2$ are expressed by the roots of the cubic equation

$$u^3 - Pu^2 + Qu - R = 0 \qquad (2),$$

where P, Q, R involve the roots of the proposed equation (1) *symmetrically*, and are therefore expressible in terms of its coefficients and are consequently known.

Inasmuch as

$$A_1 = 2(\rho + \rho_2)(\rho_1 + \rho_3)$$

$$= 2(\rho\rho_1 + \rho\rho_3 + \rho_1\rho_2 + \rho_2\rho_3)$$

$$= 2q - 2(\rho\rho_2 + \rho_1\rho_3) = 2q - 2u',$$

where $u' = \rho\rho_2 + \rho_1\rho_3$, we may transform the equation (2) into

$$u'^3 - P'u'^2 + Q'u' - R' = 0, \qquad (3),$$

when $P' = q$, $Q' = pr - 4s$, $R' = (p^2 - 4q)s + r^2$ are assigned by the process given in Art. 981 *.

We thus find $A_1 = 2q - 2u'$, and therefore

$$\theta_1 = p^2 - 2A_1 = p^2 - 4q + 4u':$$

if we find a real value of u', or of θ_1, we get

$$X_1 = \frac{p + \surd\theta_1}{2}, \quad X_2 = \frac{p - \surd\theta_1}{2}.$$

Again, since

$$X_1 = \rho + \rho_2,$$

we may consider

$$x^2 - X_1 x + \lambda = 0$$

as a factor of the proposed equation: and if we proceed to examine the conditions (as in Art. 985) which this factor must satisfy, we shall readily find

$$\lambda = \frac{X_1^3 - pX_1^2 + qX_1 - r}{2X_1 - p}:$$

we thus finally get, as in Art. 1021,

$$\rho = \frac{X_1 + \surd(X_1^2 - 4\lambda)}{2} \text{ and } \rho_2 = \frac{X_1 - \surd(X_1^2 - 4\lambda)}{2},$$

and similarly for ρ_1 and ρ_3.

Theory of Euler's solution of a biquadratic equation.

1028. If instead of forming the equation which expresses the triple values of A_1, we had begun at once to form the equation which expresses the triple values of θ, we should have

* The processes of solution of biquadratic equations, which we have considered in Chap. XLII., are equivalent to the corresponding processes for forming the reducing cubic equations, whose roots are such *symmetrical* combinations of those of the proposed equation as have triple values only: such are $(\rho + \rho_2)(\rho_1 + \rho_3)$ or $(\rho\rho_2 + \rho_1\rho_3)$, or $(\rho + \rho_2 - \rho_1 - \rho_3)^2$, and others which may be formed: such processes are always greatly simplified, by supposing the coefficient of the second term of the biquadratic equation to be zero: thus the equation, whose roots are $(\rho + \rho_2 - \rho_1 - \rho_3)^2$, investigated in the Article which follows, when all the terms of the proposed equation are complete, is

$$u^3 - (3p^2 - 8q)u^2 + (3p^4 - 16p^2q + 16pr + 16q^2 - 64s)u - (p^3 - 4pq + 8r)^2 = 0,$$

which becomes, when $p = 0$,

$$u^3 + 8qu^2 + (16q^2 - 64s) - 64r^2 = 0:$$

it is only in this latter form that the particular process of transformation (if so it may be termed) which is commonly known as Euler's solution (Art. 986), is practicable: and it is to this form of the original equation *exclusively* that Des Cartes' solution (Art. 985) is capable of application.

found

$$p = \rho + \rho_2 + \rho_1 + \rho_3,$$

$$t = \rho + \rho_2 - \rho_1 - \rho_3,$$

from whence we should obtain

$$\tfrac{1}{2}(p+t) = \rho + \rho_2, \quad \text{and} \quad \tfrac{1}{2}(p-t) = \rho_1 + \rho_3:$$

if we farther make $u + v = \rho\rho_2$, and $u - v = \rho_1\rho_3$, the two quadratic factors of the proposed equation would become

$$x^2 - \tfrac{1}{2}(p+t)\,x + u + v = 0,$$

$$x^2 - \tfrac{1}{2}(p-t)\,x + u - v = 0,$$

which multiplied together would give the equation

$$x^4 - px^3 + \{2u + \tfrac{1}{4}(p^2 - t^2)\}\,x^2 - (pu + tv)\,x + u^2 - v^2 = 0:$$

if we equate the terms of this biquadratic equation with those of the equation proposed, we get

$$2u + \tfrac{1}{4}(p^2 - t^2) = q,$$

$$pu + tv = r,$$

$$u^2 - v^2 = s.$$

If from these three equations, we eliminate u and v by the ordinary methods, and replace t^2 by θ, we get the cubic equation sought for, which is

$$\theta^3 - (3p^2 - 8q)\,\theta^2 + (3p^4 - 16p^2q + 16pr + 16q^2 - 64s)\,\theta$$
$$- (p^3 - 4pq + 8r)^2 = 0.^*$$

If θ_1, θ_2, θ_3 be the three values of θ in this equation, we get

$$\rho = \frac{p + \sqrt{\theta_1} + \sqrt{\theta_2} + \sqrt{\theta_3}}{4},$$

$$\rho_1 = \frac{p - \sqrt{\theta_1} + \sqrt{\theta_2} - \sqrt{\theta_3}}{4},$$

$$\rho_2 = \frac{p + \sqrt{\theta_1} - \sqrt{\theta_2} - \sqrt{\theta_3}}{4},$$

$$\rho_4 = \frac{p - \sqrt{\theta_1} - \sqrt{\theta_2} + \sqrt{\theta_3}}{4}.$$

* Ivory. Article "Equations" in the Encyclopædia Britannica.

It will be observed that the last term of the equation which expresses the values of θ with its sign changed (Art. 1027, Note) or

$$\theta_1 \times \theta_2 \times \theta_3 = (p^3 - 4pq + 8r)^2,$$

and therefore

$$\sqrt{\theta_1} \times \sqrt{\theta_2} \times \sqrt{\theta_3} = \pm (p^3 - 4pq + 8r).$$

If we suppose $\rho_1 = 0$, $\rho_2 = 0$, $\rho_3 = 0$, and therefore $\rho = p$, then $\sqrt{\theta_1} \times \sqrt{\theta_2} \times \sqrt{\theta_3} = \rho^3 = p^3$, which shews that the upper or positive sign must be taken: if we suppose therefore $p^3 - 4pq + 8r$ to be a positive quantity, then either all the values of $\sqrt{\theta_1}$, $\sqrt{\theta_2}$, $\sqrt{\theta_3}$ must be positive, or they must become negative by pairs, as in the form of solution just given: but if $p^3 - 4pq + 8r$ be a negative quantity, then either all the values of $\sqrt{\theta_1}$, $\sqrt{\theta_2}$, and $\sqrt{\theta_3}$ must be negative or two of them positive, in which case the solution would assume the following form:

$$\rho = \frac{p - \sqrt{\theta_1} - \sqrt{\theta_2} - \sqrt{\theta_3}}{4},$$

$$\rho_1 = \frac{p + \sqrt{\theta_1} - \sqrt{\theta_2} + \sqrt{\theta_3}}{4},$$

$$\rho_2 = \frac{p - \sqrt{\theta_1} + \sqrt{\theta_2} + \sqrt{\theta_3}}{4},$$

$$\rho_3 = \frac{p + \sqrt{\theta_1} + \sqrt{\theta_2} - \sqrt{\theta_3}}{4},$$

Failure of this method of solution when applied to an equation of 5 or higher dimensions.

1029. If we should proceed to apply the same method to the expression of the roots of the equation

$$x^5 - px^4 + qx^3 - rx^2 + sx - s = 0,$$

we should find

$$t = \rho + \alpha\rho_1 + \alpha^2\rho_2 + \alpha^3\rho_3 + \alpha^4\rho_4,$$

where α, α^2, α^3, α^4 are the roots of the equation

$$x^5 - 1 = 0,$$

which are different from 1: if we now make $\theta = t^5$, we should obtain an equation in θ, with determinate coefficients of $1 \times 2 \times 3 \times 4$ dimensions, which is decomposable into 2×3 biquadratic equations of the form

$$\theta^4 - P\theta^3 + Q\theta^2 - R\theta + S = 0,$$

where each of the coefficients P, Q, R, S admits of 2×3 (Art. 1024) different values for different permutations of the roots: or in other words, those coefficients will be dependent upon the solutions of equations of the sixth degree: and it appears, therefore, that the solution of an equation of the fifth degree, will necessarily lead, *by the process under consideration*, to an equation of the sixth degree.

1030. If we apply the preceding process, however, to the solution of the binomial equation

Application of this theory to binomial equations.

$$x^n - 1 = 0$$

we shall find that it will enable us to assign its roots, by algebraical processes alone, for *all* values of n.

For this purpose, let us suppose n to be a *prime* number, and let us represent the roots of

Formation of a cyclical period of the roots when n is a prime number.

$$\frac{x^n - 1}{x - 1} = 0$$

by $r, r^2, r^3 \dots r^{n-2}$, and let us suppose them to be further distributed into the *cyclical* period

$$r, \; r^a, \; r^{a^2}, \; r^{a^3} \dots \; r^{a^{n-2}},$$

by means of any one of the primitive roots (Art. 531) of n, such as a: we shall thus find that the substitution of r^a, r^{a^2}, r^{a^3}, &c, or of any one of its terms, in the place of r, will reproduce the same series in the same order, if no regard be paid to the position of the first term, or, in other words, if they be supposed to be arranged in a circular order (Art. 722).

1031. Assuming therefore α to be a root of

Determination of the roots of the reducing and from thence of the primitive equation.

$$x^{n-1} - 1 = 0,$$

which is different from 1, we may make

$$t = r + \alpha r^a + \alpha^2 r^{a^2} + \dots \alpha^{n-2} r^{a^{n-2}},$$

and therefore

$$\theta = t^{n-1} = A_0 + \alpha A_1 + \alpha^2 A_2 + \dots \alpha^{n-2} A_{n-2},$$

where $A_0, A_1, A_2 \dots A_{n-2}$ are rational expressions involving r, which do not change upon the substitution of r^a, r^{a^2}, r^{a^3}, &c.

for r: for if we replace t by $\alpha^{n-2}t,\ \alpha^{n-3}t \dots \alpha t$, successively, we get

$$\alpha^{n-2}t = \alpha^{n-2}r + r^{a} + \alpha r^{a^2} + \dots\dots \alpha^{n-3} r^{a^{n-2}},$$

$$\alpha^{n-3}t = \alpha^{n-3}r + \alpha^{n-2} r^{a} + r^{a^2} + \dots \alpha^{n-4} r^{a^{n-2}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\alpha t = \alpha r + \alpha^2 r^{a} + \alpha^3 r^{a^2} + \dots\dots \quad r^{a^{n-2}};$$

and if again in the same expression for t, we substitute for r the successive terms of the series $r^{a},\ r^{a^2},\ r^{a^3} \dots r^{a^{n-2}}$, we shall get

$$r^{a} + \alpha r^{a^2} + \alpha^2 r^{a^3} + \ \dots\ \alpha^{n-2} r = \alpha^{n-2} t,$$

$$r^{a^2} + \alpha r^{a^3} + \alpha^2 r^{a^4} + \dots \alpha^{n-2} r^{a} = \alpha^{n-3} t,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$r^{a^{n-2}} + \alpha r + \alpha^2 r^{a} + \ \dots\ \alpha^{n-2} r^{a^{n-3}} = \alpha t;$$

it thus appears that the values of θ or of

$$t^{n-1} = (\alpha t)^{n-1} = (\alpha^2 t)^{n-1} = \dots (\alpha^{n-2} t)^{n-1}$$

are the same, whatever be the term of the series

$$r^{a},\ r^{a^2},\ r^{a^3} \dots r^{a^{n-2}},$$

which we substitute for r, and that consequently those values admit of absolute determination, inasmuch as it thus appears that they involve the roots $r,\ r^{a},\ r^{a^2} \dots r^{a^{n-2}}$ symmetrically.

If we represent therefore the values of A_0 in the expression

$$\theta = A_0 + \alpha A_1 + \dots \alpha^{n-2} A_{n-2},$$

by

$$P + Q r + Q_1 r^2 + \dots Q_{n-2} r^{n-2} \qquad (1),$$

it will remain unaltered, in conformity with the proposition just demonstrated, if we replace r by r^{a}, when it becomes

$$P + Q_0 r^{a} + Q_1 r^{a^2} + \dots Q_{n-2} r \qquad (2),$$

and therefore equating corresponding terms of the identical expressions (1) and (2), we get

$$Q = Q_0, \quad Q_1 = Q_2, \quad Q_2 = Q_3 \dots Q_{n-3} = Q_{n-2},$$

and therefore

$$A_0 = P + Q\,(r + r^{a} + \dots r^{a^{n-2}}),$$

$$= P - Q,$$

since

$$r + r^{a} + r^{a^2} + \dots r^{a^{n-2}} = -1:$$

in a similar manner we find

$$A_1 = P_1 - Q_1,$$

$$A_2 = P_2 - Q_2,$$

$$A_{n-2} = P_{n-2} - Q_{n-2},$$

where the symbols $P, P_1 \dots P_{n-2}, Q, Q_1 \dots Q_{n-2}$ denote rational numbers.

The values of θ are therefore

$$\theta_0 = P - Q + (P_1 - Q_1) + (P_2 - Q_2) + \dots (P_{n-2} - Q_{n-2}),$$

$$\theta_1 = P - Q + \alpha_1 (P_1 - Q_1) + \alpha_1^2 (P_2 - Q_2) + \dots \alpha_1^{n-2} (P_{n-2} - Q_{n-2}),$$

$$\theta_2 = P - Q + \alpha_2 (P_1 - Q_1) + \alpha_2^2 (P_2 - Q_2) + \dots \alpha_2^{n-2} (P_{n-2} - Q_{n-2}),$$

$$\theta_{n-2} = P - Q + \alpha_{n-2} (P_1 - Q_1) + \alpha_{n-2}^2 (P_2 - Q_2) + \dots \alpha_{n-2}^{n-2} (P_{n-2} - Q_{n-2}),$$

where $\alpha_1 = \alpha$, $\alpha_2 = \alpha^2$, $\alpha_3 = \alpha^3 \dots \alpha_{n-2} = \alpha^{n-2}$.

We thus find

$$r = \frac{\sqrt[n-1]{\theta_0} + \sqrt[n-1]{\theta_1} + \sqrt[n-1]{\theta_2} + \dots \sqrt[n-1]{\theta_{n-2}}}{n-1},$$

$$r^{a} = \frac{\sqrt[n-1]{\theta_0} + \alpha_1^{n-2} \sqrt[n-1]{\theta_1} + \alpha_2^{n-2} \sqrt[n-1]{\theta_2} + \dots \alpha_{n-2}^{n-2} \sqrt[n-1]{\theta_{n-2}}}{n-1},$$

....

$$r^{a^{n-2}} = \frac{\sqrt[n-1]{\theta_0} + \alpha_1 \sqrt[n-1]{\theta_1} + \alpha_2 \sqrt[n-1]{\theta_2} + \dots \alpha_{n-2} \sqrt[n-1]{\theta_{n-2}}}{n-1}.$$

1032. A very little consideration, however, will be sufficient to shew, that it is not necessary to form the expression for a power of θ so high as θ^{n-1}, when n is a prime number: for, under such circumstances, we have $n - 1 = mq$, and it is obvious that the terms of the expression for t may be distributed into groups of m or q terms as in Art. 1025: thus if α be a root of

Simplification of this process by making α the root, which is different from 1, of the binomial equation whose

$$x^m - 1 = 0,$$

index is the least factor of $n-1$.

which is different from 1, we have

$$t = r + \alpha r^{a} + \alpha^2 r^{a^2} + \dots\dots\dots\dots\dots\dots \alpha^{m-1} r^{a^{m-1}},$$

$$+ r^{a^m} + \alpha r^{a^{m+1}} + \alpha^2 r^{a^{m+2}} + \dots\dots\dots\dots \alpha^{m-1} r^{a^{2m-1}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ r^{a^{(q-1)m}} + \alpha r^{a^{(q-1)m+1}} + \alpha^2 r^{a^{(q-1)m+2}} + \dots \alpha^{m-1} r^{a^{qm-1}};$$

and if we assume

$$X_1 = r + r^{a^m} + r^{a^{2m}} + \dots\dots r^{a^{(q-1)m}},$$

$$X_2 = r^{a} + r^{a^{m+1}} + r^{a^{2m+1}} + \dots r^{a^{(q-1)m+1}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$X_m = r^{a^{m-1}} + r^{a^{2m-1}} + \dots\dots\dots r^{a^{qm-1}},$$

it will obviously appear, from an examination of the terms of which these several expressions are composed, that by replacing r by r^a, X_1 becomes X_2, X_2 becomes X_3, ... X_{m-2} becomes X_{m-1}: similarly, by replacing r by r^{a^2}, X_1 becomes X_3, X_2 becomes X_4 ... X_{m-2} becomes X_1: and similar consequences will follow from replacing r by other terms of the cyclical period of the roots: it will follow therefore that

$$t = X_1 + \alpha X_2 + \alpha^2 X_3 + \dots \alpha^{m-1} X_m$$

will become, when r is replaced by r^a,

$$X_2 + \alpha X_3 + \alpha^2 X_4 + \dots \alpha^{m-1} X_1 = \alpha^{m-1} t;$$

when r is replaced by r^{a^2}, it will become

$$X_3 + \alpha X_4 + \alpha^2 X_5 + \dots \alpha^{m-1} X_2 = \alpha^{m-2} t:$$

and when r is replaced by $r^{a^{m-1}}$, it will become

$$X_m + \alpha X_1 + \alpha^2 X_2 + \dots \alpha^{m-1} X_{m-1} = \alpha t.$$

It will follow, therefore, that if

$$\theta = t^m = (\alpha t)^m = (\alpha^2 t)^m = \dots (\alpha^{m-1} t)^m$$

$$= A_0 + \alpha A_1 + \alpha^2 A_2 + \dots \alpha^{m-1} A_{m-1},$$

A_0, A_1, A_2 ... A_{m-2} will be rational expressions involving X_1, X_2 ... X_m in such a manner as not to change, when X_1 is changed into X_2, X_2 into X_3, and so on: and that they are consequently of the form $P + Q(X_1 + X_2 + X_3 + \dots X_m)$ or $P + Qs$, where P and Q are determinate numbers, and s is either -1, or as will be

seen hereafter, some known rational or irrational number: if we now find therefore the different values of θ, which are

$$\theta_0 = P_0 + Q_0 s + (P_1 + Q_1 s + \ldots\ldots + (P_{m-1} + Q_{m-1} s),$$

$$\theta_1 = P_0 + Q_0 s + \alpha_1 (P_1 + Q_1 s) + \ldots + \alpha_1^{m-1} (P_{m-1} + Q_{m-1} s),$$

$$\theta_2 = P_0 + Q_0 + \alpha_2 (P_1 + Q_1 s) + \ldots\ldots + \alpha_2^{m-1} (P_{m-1} + Q_{m-1} s),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\theta_{m-1} = P_0 + Q_0 + \alpha_{m-1} (P_1 + Q_1 s) + \ldots + \alpha_{m-1}^{m-1} (P_{m-1} + Q_{m-1} s),$$

we shall find, making $\sqrt[m]{\theta_0} = s$,

$$X_1 = \frac{s + \sqrt[m]{\theta_1} + \sqrt[m]{\theta_2} + \ldots + \sqrt[m]{\theta_{m-1}}}{m},$$

$$X_2 = \frac{s + \alpha_1^{m-1} \sqrt[m]{\theta_1} + \alpha_2^{m-1} \sqrt[m]{\theta_2} + \ldots + \alpha_{m-1}^{m-1} \sqrt[m]{\theta_{m-1}}}{m},$$

$$X_3 = \frac{s + \alpha_1^{m-2} \sqrt[m]{\theta_1} + \alpha_2^{m-2} \sqrt[m]{\theta_2} + \ldots + \alpha_{m-1}^{m-2} \sqrt[m]{\theta_{m-1}}}{m},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_m = \frac{s + \alpha_1 \sqrt[m]{\theta_1} + \alpha_2 \sqrt{\theta_2} + \ldots + \alpha_{m-1} \sqrt[m]{\theta_{m-1}}}{m} *.$$

Having thus found the values of $X_1, X_2 \ldots, X_m$, it remains to determine the several roots, which are q in number, in the several groups which they form: for this purpose, we consider them as forming the roots of an equation of q dimensions, and assume

$$t = r + \alpha r^{a^m} + \alpha^2 r^{a^{2m}} + \ldots \alpha^{(q-1)} r^{a^{(q-1)m}},$$

* If $mq = n - 1$, and if $q = 2$, and $m = \frac{n-1}{2}$, we find

$$X_1 = r + r^{a^{\frac{n-1}{2}}} = r + \frac{1}{r}: \text{ (Art. 532)}$$

$$X_2 = r^a + \frac{1}{r^a},$$

$$X_3 = r^{a^2} + \frac{1}{r^{a^2}} \ldots$$

If we make therefore

$$r^m = \cos \frac{2m\pi}{n} + \sqrt{-1} \sin \frac{2m\pi}{n},$$

we find

$$X_1 = 2 \cos \frac{2\pi}{n}, \quad X_2 = 2 \cos \frac{2a\pi}{n}, \quad X_3 = 2 \cos \frac{2a^2\pi}{n} \ldots$$

where a is a root of the equation

$$x^q - 1 = 0,$$

which is different from 1: we then form

$$\theta = t^q = A_0 + a A_1 + a^2 A_2 + \ldots + a^{q-1} A_{q-1},$$

where A_0, A_1, $A_2 \ldots A_{q-1}$ are reducible to the form $P + Qs$, where P and Q are severally determinate numbers, and $s = X_1$, whose value has been previously found: the several values of θ, and consequently those of $r a^m$, $r a^{2m} \ldots r a^{(q-1)m}$ are determined, as in the last Article, merely replacing s by X_1 and m by q.

If, however, it should be found that q is still a composite number, the several roots which are comprehended in X_1, X_2, X_3, X_m may be further distributed as before, and the process in the last Article repeated: for it is always desirable to reduce as low as possible the index of the power to which t is required to be raised.

We shall now proceed to apply the preceding theory to the solution of binomial equations.

1033. Let it be required to solve the equation

$$x^5 - 1 = 0.$$

Application of the preceding theory to the solution of the equation $x^5 - 1 = 0$.

The number 2 is a *primitive* root of 5, and the *cyclical* period of the roots of this equation is therefore

$$r,\ r^2,\ r^{2^2},\ r^{2^3},$$

or

$$r,\ r^2,\ r^4,\ r^3.$$

In this case, since $4 = (n-1) = 2 \times 2 = qm$, if we make $a = -1$ or that root of $x^2 - 1 = 0$, which is different from 1, we find

$$t = X_1 + a X_2,$$

where $X_1 = r + r^4$ and $X_2 = r^2 + r^3$.

Consequently

$$\theta = t^2 = A_0 + a A_1,$$

where

$$A_1 = 2 X_1 X_1 = 2 (r + r^2 + r^3 + r^4) = -2:$$

and $A_0 = s^2 - A_1$ (Art. 1022) $= 3$: therefore

$$\theta_1 = 3 - 2a = 5.$$

We thus get, since $\sqrt{\theta_0} = s = -1$,

$$X_1 = \frac{\sqrt{\theta_0} + \sqrt{\theta_1}}{2} = \frac{-1 + \sqrt{5}}{2},$$

$$X_2 = \frac{\sqrt{\theta_0} + \alpha\sqrt{\theta_1}}{2} = \frac{-1 - \sqrt{5}}{2}.$$

But inasmuch as

$$X_1 = r + r^4 \text{ and } X_2 = r^2 + r^3,$$

we find, by repeating the same process,

$$t' = r + \alpha r^4,$$

and therefore

$$\theta' = t'^2 = r^2 + r^3 + 2\alpha = X_2 + 2\alpha = X_2 - 2.$$

We thus get, since $\sqrt{\theta'_0} = X_1$,

$$r = \frac{X_1 + \sqrt{\theta'_1}}{2} = \frac{X_1 + \sqrt{(X_2 - 2)}}{2},$$

$$r^4 = \frac{X_1 + \alpha\sqrt{\theta'_1}}{2} = \frac{X_1 - \sqrt{(X_2 - 2)}}{2}:$$

and again, by changing X_1 into X_2, and X_2 into X_1, we find

$$r^2 = \frac{X_2 + \sqrt{(X_1 - 2)}}{2},$$

$$r^3 = \frac{X_2 - \sqrt{(X_1 - 2)}}{2}.$$

If we replace X_1 and X_2 by their numerical values, we get

$$r = \frac{-1 + \sqrt{5} + \sqrt{(-10 - 2\sqrt{5})}}{4},$$

$$r^4 = \frac{-1 + \sqrt{5} - \sqrt{(-10 - 2\sqrt{5})}}{4},$$

$$r^2 = \frac{-1 - \sqrt{5} + \sqrt{(-10 + 2\sqrt{5})}}{4},$$

$$r^3 = \frac{-1 - \sqrt{5} - \sqrt{(-10 - 2\sqrt{5})}}{4}.$$

These will be found to coincide with the values given in Arts. 707 and 814.

Solution of the equation $x^7 - 1 = 0$.

1034. Let it be required to solve the equation

$$x^7 - 1 = 0.$$

Since 3 is a primitive root of 7, the cyclical period of the roots of the equation

$$\frac{x^7 - 1}{x - 1} = 0$$

is

$$r,\ r^3,\ r^2,\ r^6,\ r^4,\ r^5.$$

Again, since $n - 1 = 6 = 3 \times 2 = qm$, if we take α the root of

$$\frac{x^2 - 1}{x - 1} = 0,$$

we get

$$t = X_1 + \alpha X_2,$$

where

$$X_1 = r + r^2 + r^4,$$

$$X_2 = r^3 + r^6 + r^5.$$

Therefore

$$\theta = t^2 = A_0 + \alpha A_1,$$

where

$$A_1 = 2 X_1 X_2 = 2(3 + r + r^3 + r^2 + r^6 + r^4 + r^5)$$

$$= 2(3 + s) = 6 + 2s = 4, \text{ since } s = -1,$$

and therefore

$$A_0 = s^2 - A_1 = 1 - 4 = -3.$$

We thus find

$$\theta = -3 + 4\alpha,$$

and therefore

$$\theta_1 = -7.$$

Consequently

$$X_1 = \frac{s + \sqrt{\theta_1}}{2} = \frac{-1 + \sqrt{-7}}{2},$$

$$X_2 = \frac{s - \sqrt{\theta_1}}{2} = \frac{-1 - \sqrt{-7}}{2}.$$

Again, since $q = 3$, if we make α a root of the equation

$$\frac{x^3 - 1}{x - 1} = 0,$$

and consider r, r^2, r^4, which are involved in X_1, as the roots of a new equation, we shall find

$$t' = r + \alpha r^2 + \alpha^2 r^4,$$

and therefore

$$\theta' = t'^3 = A_0 + \alpha A_1 + \alpha^2 A_2,$$

where by the actual formation of the cube of t', we find

$$A_0 = 6 + r^3 + r^6 + r^5 = 6 + X_2,$$

$$A_1 = 3(r + r^2 + r^4) = 3X_1,$$

$$A_2 = 3(r^3 + r^6 + r^5) = 3X_2,$$

and therefore

$$\theta' = 6 + X_2 + 3\alpha X_1 + 3\alpha^2 X_2.$$

We thus get

$$\theta'_1 = 7 - \sqrt{7}\left(\frac{3\sqrt{3} + \sqrt{-1}}{2}\right),$$

$$\theta'_2 = 7 - \sqrt{7}\left(\frac{3\sqrt{3} - \sqrt{-1}}{2}\right):$$

and therefore

$$r = \frac{X_1 + \sqrt[3]{\theta'_1} + \sqrt[3]{\theta'_2}}{3},$$

$$r^2 = \frac{X_1 + \alpha^2\sqrt[3]{\theta'_1} + \alpha\sqrt[3]{\theta'_2}}{3},$$

$$r^4 = \frac{X_1 + \alpha\sqrt[3]{\theta'_1} + \alpha^2\sqrt[3]{\theta'_2}}{3}.$$

The values of r^3, r^5, and r^6 may be found in a similar manner.

1035. Let it be required to solve the equation

Solution of the equation $x^{17} - 1 = 0$.

$$x^{17} - 1 = 0.$$

Since 3 is a primitive root of 17, the *cyclical* period of the roots of

$$x^{17} - 1 = 0$$

is

$$r,\ r^3,\ r^9,\ r^{10},\ r^{13},\ r^5,\ r^{15},\ r^{11},$$

$$r^{16},\ r^{14},\ r^3,\ r^7,\ r^4,\ r^{12},\ r^2,\ r^6.$$

Since $n-1=16=8\times 2=qm$, we make α the root of

$$\frac{x^2-1}{x-1}=0,$$

and assume

$$t=X_1+\alpha X_2,$$

where

$$X_1=r+r^9+r^{13}+r^{15}+r^{16}+r^8+r^4+r^2:$$

$$X_2=r^3+r^{10}+r^5+r^{11}+r^{14}+r^7+r^{12}+r^6.$$

Therefore

$$\theta=A_0+\alpha A_1,$$

where

$$A_1=2X_1X_2=8s=-8,$$

and

$$A_0=s^2-A_1=1+8=9.$$

We thus get

$$\theta_1=9+8=17,$$

and therefore

$$X_1=\frac{-1+\surd\theta_1}{2}=\frac{-1+\surd 17}{2}=\quad 1.5615528,$$

$$X_2=\frac{-1-\surd\theta_1}{2}=\frac{-1+\surd 17}{2}=-2.5615528 \text{ *}.$$

* Generally, if n be a prime number of the form $4m+1$, and if the cyclical period formed by the roots of

$$\frac{x^n-1}{x-1}=0,$$

be distributed into two groups X_1 and X_2 of $2m$ roots each, formed by taking its alternate terms; then if we consider X_1 and X_2 as the roots of the quadratic equation

$$u^2+Au+B=0,$$

we shall find

$$A=(X_1+X_2)=-s=-1,$$

$$B=X_1X_2=ms=-\frac{(n-1)}{4}:$$

for there are $(2m)^2=4m\times m=\frac{(n-1)}{4}(n-1)$ terms in their product, forming $\frac{n-1}{4}$ original cyclical periods, and therefore equal to $\frac{n-1}{4}\times s=-\frac{(n-1)}{4}$; the equation consequently becomes

$$u^2+u-\frac{n-1}{4}=0,$$

and therefore

Again, since $q = 8 = 4 \times 2 = q'm$, we proceed to distribute the roots in the primary groups X_1 and X_2, into the secondary groups Y_1, Y_2, Y_3, Y_4 assuming

$$t' = Y_1 + \alpha Y_2,$$

where

$$Y_1 = r + r^{13} + r^{16} + r^4,$$

$$Y_2 = r^9 + r^{15} + r^8 + r^2.$$

We thus get

$$\theta' = t'^2 = A'_0 + \alpha A'_1,$$

and therefore

$$u = \frac{-1 \pm \sqrt{n}}{2}.$$

Thus if $n = 5$, we find

$$u = \frac{-1 \pm \sqrt{5}}{2}.$$

If $n = 13$, we find

$$u = \frac{-1 \pm \sqrt{13}}{2}.$$

But if n be of the form $4m + 3$, then if X_1 and X_2 be assumed as before, and be supposed to be the roots of the equation

$$u^2 + Au + B = 0,$$

we shall find

$$A = -(X_1 + X_2) = -s = 1,$$

and

$$B = X_1 X_2 = \tfrac{1}{4}(n + 1);$$

for there are $(2m + 1)^2$ or $m(n - 1) + 2m + 1$ terms in their product, forming m complete cyclical periods of $n - 1$ terms, and $(2m + 1)$ terms which are severally equal to r^n or 1; we thus get

$$B = ms + 2m + 1 = -m + 2m + 1$$

$$= m + 1 = \frac{n + 1}{4};$$

the equation thus becomes

$$u^2 + u + \frac{n + 1}{4} = 0,$$

whose roots are $\dfrac{-1 \pm \sqrt{-n}}{2}$.

Thus, if $n = 7$, we find (Art. 1034)

$$u = \frac{-1 + \sqrt{-7}}{2}.$$

If $n = 11$, we find

$$u = \frac{-1 + \sqrt{-11}}{2}.$$

This proposition is given in Gauss' *Disquisitiones Arithmeticæ*, Art. 356.

where

$$A'_1 = 2\,Y_1\,Y_2 = 2\,s = -2,$$

and

$$A'_0 = X_1^2 - A'_1 = \frac{13 - \sqrt{17}}{2}.$$

Therefore

$$\theta'_1 = \frac{13 - \sqrt{17}}{2} + 2 = \frac{17 - \sqrt{17}}{2},$$

and also

$$Y_1 = \frac{X_1 + \sqrt{\theta'_1}}{2} = \frac{-1 + \sqrt{17} + \sqrt{(34 - 2\sqrt{17})}}{4} = 2.049481.$$

$$Y_2 = \frac{X_1 - \sqrt{\theta'_1}}{2} = \frac{-1 + \sqrt{17} - \sqrt{(34 - 2\sqrt{17})}}{4} = - \;.487928.$$

In a similar manner we find

$$Y_3 = \frac{-1 - \sqrt{17} + \sqrt{(34 + 2\sqrt{17})}}{4} = \quad .344151,$$

$$Y_4 = \frac{-1 - \sqrt{17} - \sqrt{(34 + 2\sqrt{17})}}{4} = -2.905704.$$

Again, since $q' = 4 = 2 \times 2 = q''m$, we proceed to distribute the secondary groups Y_1, Y_2, Y_2, Y_3 into ternary groups Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8: taking the same value of α, we assume

$$t'' = Z_1 + \alpha Z_2,$$

where

$$Z_1 = r + r^{16} = r + \frac{1}{r},$$

$$Z_2 = r^{13} + r^4 = r^{13} + \frac{1}{r^{13}}.$$

We thus get

$$\theta'' = t''^2 = A''_0 + \alpha A''_1,$$

where

$$A''_1 = 2\,Z_1\,Z_2 = 2\,(r^3 + r^5 + r^{14} + r^{12}) = 2\,Y_3,$$

and

$$A''_0 = Z_1^2 + Z_2^2 = 4 + Y_2\,*.$$

* The value of A''_0 is more readily found from $Z_1^2 + Z_2^2$ than from $Y_1^2 - A''_1$.

Therefore

$$\theta''_1 = 4 + Y_2 - 2Y_3,$$

and

$$Z_1 = \frac{Y_1 + \sqrt{\theta''_1}}{2}$$

$$= \tfrac{1}{8}(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}}) - \tfrac{1}{4}\sqrt{(17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})},$$

$$= 1.864944,$$

$$Z_2 = \frac{Y_1 - \sqrt{\theta''_1}}{2} = .184537.$$

In a similar manner, we find

$$Z_3 = r^9 + \frac{1}{r^9} = -1.965946,$$

$$Z_4 = r^{15} + \frac{1}{r^{15}} = 1.478018,$$

$$Z_5 = r^3 + \frac{1}{r^3} = .891477,$$

$$Z_6 = r^5 + \frac{1}{r^5} = -.547326,$$

$$Z_7 = r^{10} + \frac{1}{r^{10}} = -1.700434,$$

$$Z_8 = r^{11} + \frac{1}{r^{11}} = -1.205269.$$

The roots of the equation $u^2 - Z_1 u + 1 = 0$, or r and r^{16}, are $.932472 \pm .261124\sqrt{-1}$.

The roots of the equation $u^2 - Z_4 u + 1 = 0$, or r^2 and r^{15}, are $.739009 \pm .673696\sqrt{-1}$.

The roots of the equation $u^2 - Z_5 u + 1 = 0$, or r^3 and r^{14}, are $.445738 \pm .895163\sqrt{-1}$.

The roots of the equation $u^2 - Z_2 u + 1 = 0$, or r^4 and r^{13}, are $.092368 \pm .995734\sqrt{-1}$.

The roots of the equation $u^2 - Z_6 u + 1 = 0$, or r^5 and r^{12}, are $-.273662 \pm .961183\sqrt{-1}$.

The roots of the equation $u^2 - Z_8 u + 1 = 0$, or r^6 and r^{11}, are $-.602635 \pm .798017 \sqrt{-1}$.

The roots of the equation $u^2 - Z_7 u + 1 = 0$, or r^7 and r^{10}, are $-.850217 \pm .526432 \sqrt{-1}$.

The roots of the equation $u^2 - Z_3 u + 1 = 0$, or r^8 and r^9, are $-.982973 \pm .183749 \sqrt{-1}$.

We have thus assigned all the roots of the equation

$$x^{17} - 1 = 0$$

without the solution of any equation which exceeds the second degree.

A regular figure of 17 sides may be inscribed geometrically in a circle.

1036. The solution of the equation, $x^{17} - 1 = 0$, which we have given in the preceding Article, enables us to divide a circle geometrically into 17 equal parts: for we have elsewhere shewn (Art. 811) that r, which is expressed by

$$.932472 + .261124 \sqrt{-1},$$

is equally represented in magnitude and direction by

$$\cos \frac{2\pi}{17} + \sqrt{-1} \sin \frac{2\pi}{17},$$

being a radius of the circle which is equal in magnitude to 1, and which makes an angle of $\frac{2\pi}{17}$ with the primitive line.

The processes which have been employed for determining the numerical values of $\cos \frac{2\pi}{17}$ and $\sin \frac{2\pi}{17}$, in the preceding value of r, involve nothing beyond the extraction of the square roots of real magnitudes and the solution of quadratic equations with real coefficients, which are equally within the province of arithmetic and of geometry.

Other regular figures which possess the same property.

1037. Whenever n is a prime number, and $n - 1 = 2^m$, the equation

$$x^n - 1 = 0$$

may be similarly solved by the aid of quadratic equations with real terms, and the subdivision of the circle into n or $2^m + 1$ equal parts may be effected by arithmetical and geometrical means: but a very little consideration will be sufficient to shew that $2^m + 1$ cannot be a prime number, unless m is of the form

2^{r} *, and not universally † so, even when that condition is fulfilled: we thus find the series of numbers

$$2^1+1,\quad 2^2+1,\quad 2^{2^2}+1,\quad 2^{2^3}+1,\quad 2^{2^4}+1,\ldots$$

or

$$3,\quad 5,\quad 17,\quad 257,\quad 65537\ldots$$

which express the number of sides of regular figures which are geometrically inscriptible in a circle.

More generally it follows from the preceding theory "that, if n be a prime number, and $n-1=a^{\lambda}\times b^{\mu}\times c^{\nu}\ldots$, then the binomial equation

$$x^n-1=0$$

may be solved, and a circle divided into n equal parts, by processes which involve the solution of λ equations of a dimensions, of μ equations of b dimensions, of ν equations of c dimensions..., and all such subordinate equations admit of resolution by general processes which fail when applied to other equations in their general form of a degree superior to the fourth." Thus if $n=7$, we find $n-1=2\times 3$, and a regular heptagon may be inscribed in a circle by the solution of a quadratic and cubic equation: if $n=13$, we have $n=2^2\times 3$, and a regular polygon of 13 sides may be inscribed by the solution of two quadratic and of one cubic equation. It is this important proposition which forms the subject of the 7th and concluding section of the *Disquisitiones Arithmeticæ* ‡ of Gauss, and which may be considered as one of the most remarkable additions which the theory of equations has received in later times.

* For if $m=qr$, where q is an odd number, we get $2^m=2^{qr}+1$ which is divisible by 2^r+1: for if we make $2^r=u$, we get $2^{qr}+1=u^q+1$, which is divisible by 2^r+1 or $u+1$, since q is an odd number; but 2^n is the only form of m which contains no factor which is an odd number.

† For $2^{2^{32}}+1$ or the 32nd term of the series, has been shewn by Euler to be divisible by 641; all the terms in the series which precede it are prime numbers.

‡ The same subject is treated in a very general form by Abel in a very remarkable "*Memoire sur une classe particulière d'equations resolubles algebriquement*," which appears in the first volume of his works, p. 114, and of which a short analysis is given in a "Report on the recent progress and present state of certain branches of Analysis," p. 318, amongst the Reports of the British Association for 1833.

CHAPTER XLVI.

SOURCES OF AMBIGUITY.

In Arithmetical Algebra, the results of direct and inverse operations are equally unique.

1038. In Arithmetical Algebra, the results of direct as well as of inverse operations, when practicable, are equally unique, the only ambiguities being those which correspond to different cases, admitting of distinct and independent statements: thus the only square of $1+x$ is $1+2x+x^2$, and the only recognized square root of $1+2x+x^2$ is $1+x$: the only square of $(1-x)$ is $1-2x+x^2$, and the only recognized square root of $1-2x+x^2$ is $1-x$, it being assumed that the square and the root are equally arithmetical both in their value and in their arrrangement: and though the arithmetical values of $1-2x+x^2$ and x^2-2x+1 are the same, yet their roots, being $1-x$ in one case, and $x-1$ in the other, are different both in their values and their arrangement: and it is only when $1-2x+x^2$ and x^2-2x+1 are derived in such a manner that one of them may replace the other indifferently, which can only take place when the relations of value of 1 and x are unknown, that the square roots of either one of the two, or of both, will equally admit of being considered as real and arithmetical results.

In Arithmetical Algebra, ambiguities can only present themselves when the relations of the values of the symbols connected with them are unknown.

1039. Ambiguities, therefore, can only present themselves in Arithmetical Algebra, when the specific values of the symbols are not assigned antecedently to the performance of operations upon them: for it is only under such circumstances that expressions, such as $1-2x+x^2$ and x^2-2x+1, can be considered to be equally admissible as the subjects of any operation, such as that of extracting the square root, or that $1-x$, which is the square root of the first of them, or $x-1$ which is the square root of the second, can be considered as equally admissible results of the operations when performed: for it is obvious that there exists no reason, whilst the relation of those values is unknown, for the selection of one of these forms in preference to the other.

The following problems will serve to illustrate not merely the origin of such ambiguities, but likewise to suggest considerations by which they may sometimes be removed.

1040. By selling sugar at 9*d.* per lb. I lose as much per cent. as the sugar cost me. What was the prime cost of the sugar*? An ambiguous problem.

If x be the prime cost of the sugar, the conditions of the problem give us the equation

$$\frac{x-9}{x}=\frac{x}{100},$$

or

$$100x-900=x^2,$$

or

$$100x-x^2=900.$$

Subtracting both sides from 50^2 or 2500, we get

$$2500-100x+x^2=1600,$$

or

$$50-x=40,$$

which gives

$$x=10,$$

a value which answers the conditions of the equation.

If, however, in the absence of any knowledge of the value of x, or of the limits within which it might be taken, we should reverse the order of the terms in the trinomial $2500-300x+x^2$, we should find

$$x^2-100x+2500=1600,$$

or

$$x-50=40,$$

which gives $x=90$, a number which equally answers the conditions of the problem.

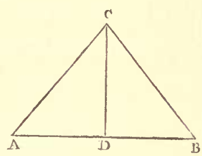
It appears, therefore, that the problem proposed is really ambiguous, and that the processes of Arithmetical Algebra equally adapt themselves to the determination of the two values of the unknown symbol which is required to be found. Such an ambiguity in this and in similar cases could only be made to disappear, by supposing the range of its values to be limited by

* See a similar problem, No. 7, Art. 409.

circumstances involved in the statement of the problem in such a manner as to be incompatible with one or both of the two values which are thus assigned.

A problem in which the limitations of value of the unknown quantity remove the ambiguity which would otherwise appear.

1041. Thus, suppose it was required to determine a right-angled triangle, whose hypothenuse was given, as well as the sum of the other two sides and of the perpendicular let fall from the right angle upon the hypothenuse: if we suppose the triangle to be ACB, and if $AB = b$, $AC + CB + CD = a$, and $CD = x$: and if we further assume $AC - CB = y$, we should find



$$AC = \frac{a - x + y}{2}, \quad CB = \frac{a - x - y}{2},$$

and therefore, since $AC^2 + CB^2 = AB^2$,

$$\frac{a^2 - 2ax + x^2 + y^2}{2} = b^2 \qquad (1).$$

Again, since $AC \times CB = AB \times CD$, we get

$$\frac{a^2 - 2ax + x^2 - y^2}{4} = bx \qquad (2).$$

If we eliminate y^2 from the equations (1) and (2), we get the final equation

$$2ax + 2bx - x^2 = a^2 - b^2,$$

one solution of which gives us

$$x = a + b - \sqrt{(2ab + 2b^2)},$$

a positive value and which answers the conditions of the problem proposed.

If however, in order to find the second value of x, we should arrange the equation under the form

$$x^2 - 2(a + b)x + (a + b)^2 = 2ab + 2b^2,$$

we should not be authorized, by the principles of Arithmetical Algebra, in forming the root

$$x - (a + b) = \sqrt{(2ab + 2b^2)},$$

inasmuch as we know, from the statement of the question, that

x is less than a, and therefore than $a+b$, and that consequently the arrangement of the terms of the square

$$x^2 - 2(a+b)x + (a+b)^2,$$

and therefore, its root, is not arithmetical: there is consequently only one root of the proposed equation, which is compatible with the conditions of the problem in which it originates.

If, however, we should solve the equation by the principles of Symbolical Algebra, we should find two positive roots expressed by the formula, which are

Its symbolical solution.

$$a + b \pm \sqrt{(2ab + 2b^2)},$$

the greater of which must be rejected as incompatible with the conditions of the problem for the reasons which we have just given.

1042. The ambiguities of Symbolical, as well as those of Arithmetical, Algebra, are confined to inverse operations and in ordinary cases, in the absence of interpretation, the results which we obtain in the solution of problems, are equally unambiguous in one science and in the other: thus, if the symbolical roots of a^2 be equally $+a$ and $-a$, and if $-a$ be either uninterpreted or uninterpretable, the negative root, whenever it presents itself, may be rejected as unmeaning or inapplicable, and the conclusion restricted to that which is furnished by the positive root or roots, which are generally, though not universally, equally obtainable by the processes of Arithmetical Algebra.

Considerations by which the ambiguities of Symbolical Algebra may sometimes be removed.

Thus in the solution of the following problem: "*A* bought pears for 72*d*., and found that if he had bought six more for the same money, he would have paid 1*d*. less for each. How many did he buy?" we readily find, by Symbolical Algebra, the two roots of the resulting equation, which are 18 and -24: and inasmuch as no meaning relative to the problem, in the form in which it was proposed, can be given to the negative root, we derive no information, from the application of the principles of Symbolical Algebra, which those of Arithmetical Algebra, in such a case, would not equally supply*.

Problem.

* It would in no respect limit the generality of this conclusion, to shew that the problem, in the text, is capable, in conformity with the ordinary principles of interpretation, of being converted into another, in which the

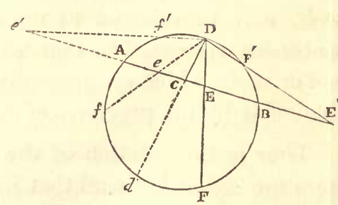
The interpretations of symbolical results must be relative to the problems in whose solutions they appear.

1043. Again, though the multiple values which Symbolical Algebra assigns for the solution of a problem, whether they be positive, negative, or imaginary, may admit of interpretation, yet they are not applicable unless they likewise satisfy the geometrical or other conditions which the problem imposes upon them: thus symbols with different signs, if applied to lines, would denote, generally, lines in opposite directions: but such opposition of direction may be very variously estimated, whether with respect to a point in the same or different lines, or with respect to another line in a given or variable position: but even supposing that lines may be found which fully satisfy the requisite symbolical conditions, yet it does not therefore follow that they are relative to the problem proposed: neither will it necessarily follow, when real ambiguities exist, that the form of the symbolical solution employed will be competent to express them, or that it will comprehend all the geometrical conditions which the problem requires to be fulfilled.

The following problems will serve to illustrate the preceding observations, as well as some other points, which deserve the most serious attention of the student.

Geometrical problem; first and imperfect solution.

1044. Let C be the middle point of a given chord AB of a circle, and let CD be drawn perpendicular to AB, meeting the circumference in D: and let it be required to draw the chord DF, so that the part of it EF intercepted between the given chord, and the circumference, may be equal to a given line.



negative root of one solution would become the positive root of the other: for this purpose, in the enunciation of the problem, we change *buying* into *selling*, *more* into *less*, *paying* into *receiving*; securing in every instance such a change of the relations of the magnitudes involved as may be correctly symbolized by a change of their sign, from + to −, and conversely; the problem thus modified, would stand as follows: "*A sold* pears for 72*d*., and found that if he had *sold* six *less* for the same money, he would have *received* 1*d*. *more* for each. How many did he *sell*?" But it should be observed that these two problems are not so related to each other, that any single interpretation of the negative root of one equation would immediately lead to the requisite modification of the other.

Let $AC = a$, $CD = b$, and the given line $EF = c$, and let us assume, in the first instance, $DE = x$: then since

$$DE \times EF = AE \times EB = AC^2 - CE^2,$$

$$= AC^2 - (DE^2 - CD^2) = AC^2 + CD^2 - DE^2,$$

we get

$$cx = a^2 + b^2 - x^2,$$

and therefore

$$x = -\frac{c}{2} \pm \sqrt{\left(\frac{c^2}{4} + a^2 + b^2\right)},$$

of which one value is positive and the other negative.

If DE be expressed by the positive value of x, the negative root will not, in this case, be expressed by a line drawn in the direction of ED produced: for the direction of the line DE is estimated with respect to that of the given line EF, which is assumed to be positive: if DE and EF, therefore, are estimated in the same direction *from* D, then they are both of them positive: but if they are reckoned or estimated in different directions from D, then EF is positive, and DE is negative: the first case is that in which $DE = -\frac{c}{2} + \sqrt{\left(\frac{c^2}{4} + a^2 + b^2\right)}$, where EF is within the circle: the second is that in which $DE' = -\frac{c}{2} - \sqrt{\left(\frac{c^2}{4} + a^2 + b^2\right)}$, where $E'F'$ is intercepted between the chord AB produced, and the circumference, and estimated in a direction opposite to DE'.

Again, the symbolical solution gives only two answers to the problem proposed, whilst the geometrical solution gives four: for it is obvious that lines Def and $Df'e'$, making with CD the same angles as DEF and $DF'E'$, will equally answer the conditions of the problem proposed.

But though the different cases formed by the pairs of symmetrical lines DEF and Def, $DF'E'$ and $Df'e'$ are distinguished from each other in the geometrical, they are not so in the algebraical, solution: and the reason of this apparent deficiency or rather ambiguity (for it is uncertain whether x expresses DE or De, DE' or De') is very easily explained: for since DE and De are equal to each other, and also symmetrical in position with respect to CD, and, therefore, when considered with reference to EF or ef, have the same sign, they

are identical in their symbolical representation; and the same remark will equally apply to DE' and De': it will follow, therefore, that the same symbols, when thus assumed and used, are ambiguous in their representation, and are incapable therefore of distinctly symbolizing more than two of the four separate cases of the problem.

Again, the four cases of the problem are reduced to two, if EF be greater than Cd or $\frac{a^2}{b}$; for in that case there are no lines DE and De drawn within the circle, which will answer the conditions of the problem: but the expression which we have obtained for DE remains unaltered in form and character, and is therefore incompetent to express the limitation to which the problem is subject: in other words, we obtain a real and positive value of x or of DE, to which no solution of the problem corresponds. It would be difficult to refer to a more instructive example of the necessity of a rigorous comparison of the symbolical values of expressions with the geometrical or other conditions which they symbolize, when the points of transition in passing from cases which are possible to those which are impossible are marked by no changes of sign or of form.

Second and more complete solution.

Such imperfections however, in the symbolical solution of geometrical problems will generally originate in an injudicious choice of the line or other quantity which the unknown symbol of the equation is assumed to represent: for if, in the case under consideration, we should represent the line CE instead of DE by x, we should easily obtain a symbolical, which is in every respect coextensive with the geometrical, solution of the problem.

For we shall thus get $DE = \sqrt{(b^2 + x^2)}$, which gives us, since $DE \times EF = AE \times EB$, the equation

$$c\sqrt{(b^2 + x^2)} = a^2 - x^2,$$

which, when rationalized, becomes

$$x^4 - (2a^2 + c^2)\,x^2 = b^2c^2 - a^4,$$

whose four roots are expressed by

$$x = \pm \sqrt{\left\{a^2 + \frac{c^2}{2} \pm c\sqrt{\left(a^2 + b^2 + \frac{c^2}{4}\right)}\right\}}.$$

It remains to examine the different values of x in this expression, and the interpretation of their meaning.

In the first place, if $\frac{a^2}{b} = c$, there are only three values of x, which are

$$0, \ \pm\sqrt{(2bc + c^2)}.$$

For, in this case, $EF = Cd$, and $CE = 0$: the points E' and e' corresponding to $\sqrt{(2bc+c^2)}$ and $-\sqrt{(2bc+c^2)}$ respectively, being exterior to the circle.

In the second place, if $\frac{a^2}{b}$ be greater than c, or if EF be less than Cd, then there are four values of x, two positive, and two negative: two of these values, which are expressed by

$$\pm\sqrt{\left\{a^2 + \frac{c^2}{2} - c\sqrt{\left(a^2 + b^2 + \frac{c^2}{4}\right)}\right\}},$$

are less than a, and therefore correspond to points E and e within the circle: and two, which are expressed by

$$\pm\sqrt{\left\{a^2 + \frac{c^2}{2} + c\sqrt{a^2 + b^2 + \frac{c^2}{4}}\right)\right\}},$$

and which are greater than a, correspond to points E' and e', which are exterior to the circle.

In the third place, if $\frac{a^2}{b}$ be less than c, or if EF be greater than Cd, then there are no points E and e within the circle which fulfil the conditions of the problem, and two of values of x, expressed by

$$\pm\sqrt{\left\{a^2 + \frac{c^2}{2} - c\sqrt{\left(a^2 + b^2 + \frac{c^2}{4}\right)}\right\}}$$

become imaginary: of the two others, expressed by

$$\pm\sqrt{\left\{a^2 + \frac{c^2}{2} + c\sqrt{\left(a^2 + b^2 + \frac{c^2}{4}\right)}\right\}},.$$

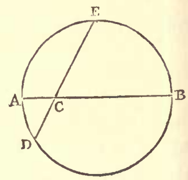
one is positive and the other negative, corresponding to points E' and e', which are exterior to the circle.

In this solution, the positive and negative roots admit of their primary and appropriate interpretation, being referred to a fixed point in a given straight line, and being estimated therefore in opposite directions from it: but the occurrence of imaginary roots intimates that there are no points E and e within the circle through which DE may be drawn so that EF may be of the

required magnitude: and it may be further observed, that there is no modified construction which the geometrical interpretation of such imaginary roots would suggest, which is compatible with the conditions of the problem*.

If the assumptions made in one part of the solution of a problem be arithmetical, they must continue so throughout.

1045. If the assumptions which are made with respect to the representation of lines or other magnitudes be arithmetical, as opposed to symbolical, in one part of the solution of a problem, the results which are obtained may be either erroneous or inconsistent, if the same principles of representation be not applied throughout. Thus, supposing a given chord DE passes through a given point C of the diameter AB of a circle, and it is required to determine the segments CD and CE into which it is divided in C; then if we should denote AC by a and CB by b, *representing lines drawn in opposite directions from* C *by symbols with the same signs*, then the same principle of representation and interpretation would be found, in the result, to extend to the lines CD and CE, which are also drawn in opposite directions from C: for if we denote CD by x, and DE by c, we should find



$$x(c - x) = ab,$$

or

$$x = \frac{c}{2} \pm \sqrt{\left(\frac{c^2}{4} - ab\right)}\dagger,$$

where the values of x are both of them positive and express the segments CD and CE of the chord: and as the assumptions made respecting them are those of Arithmetical Algebra, the results

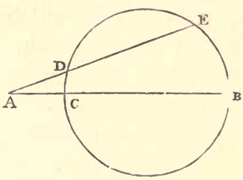
* The problem in the text, which is one of more than common instruction and interest, has been discussed, though very imperfectly, by Carnot in his *Geometrie de Position*, by whom its results are appealed to as subversive of the ordinary theory of the interpretation of positive and negative symbols, when representing lines: he appears to have laboured under the impression, in common with D'Alembert, and others who preceded him, that such interpretations of signs were absolute and not relative, and that the opposition of direction which they symbolized, was to be sought for, not in a different but in the same straight line; thus CE and Ce, in the figure in the text, would be legitimately symbolized by $+x$ and $-x$; but if DE was denoted by $+x$, the line which was symbolized by $-x$ must be sought for in the production of ED, and not in a different line DE'.

† This example is produced by D'Alembert, in his *Dissertation sur les Quantitès Negatives*, in the eighth volume of his *Opuscules*, as presenting an anomaly to the ordinary theory of interpreting negative quantities.

may be properly considered to belong to the same science and to respect quantity only and not direction.

1046. If however the point A, through which the given chord ED passes, be exterior to the circle, and if AC and AB, estimated in the same direction from A, were represented by a and b, an assumption consistent with the principles both of Symbolical and Arithmetical Algebra, and if we further assume $AD = x$ and $DE = c$, we should arrive at the equation

A problem of the same class solved both symbolically and arithmetically.



$$x\,(c + x) = a\,b,$$

and, therefore, obtain

$$x = -\frac{c}{2} \pm \sqrt{\left(\frac{c^2}{4} + a\,b\right)},$$

one value of which, representing AD is positive, and the other representing EA is negative, and exclusively belongs therefore to Symbolical Algebra: for if AD and DE estimated in the same direction from A be represented by positive symbols x and c, then EA, which is formed by the production of ED in a direction opposite to AD and DE, must, in conformity with the principles of that science, be affected with a negative sign.

If, however, the same problem be solved by the principles of Arithmetical Algebra, we may represent AD by x, AE by y, AC by a, AB by b, and DE by c, and form the two equations

$$y - x = c,$$
$$xy = a\,b.$$

If we replace y by $c + x$, the second equation becomes

$$x\,(c + x) = a\,b,$$

whose *unique* arithmetical root is

$$\sqrt{\left(\frac{c^2}{4} + ab\right)} - \frac{c}{2}:$$

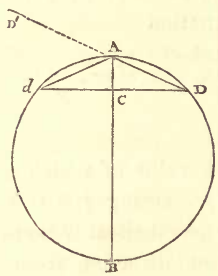
if we farther replace this value of x in the first equation, we get

$$y = c + x = \sqrt{\left(\frac{c^2}{4} + ab\right)} + \frac{c}{2},$$

which is likewise positive and arithmetical: the lines AD and AE, as well as the lines AC and AB are represented in this solution by positive symbols.

A large class of geometrical and other problems in which Symbolical Algebra affords no results which are not equally attainable by Arithmetical Algebra.

1047. In the largest class of examples, however, the equations which are formed have exclusive reference to the magnitudes of the quantities which the symbols represent, and not to their affections, or, as in the case of lines, to their positions with respect to each other: in such cases, Symbolical Algebra will give no results which are not generally obtainable by Arithmetical Algebra, inasmuch as the introduction of expressions affected with negative or imaginary signs must be considered as referrible to the general rules of Symbolical Algebra only, and not amenable therefore to the principles of interpretation which are found to be applicable under other circumstances. Thus, if we take the diameter AB of a circle, drawing Dcd perpendicular to it, and joining AD and Ad, and if we represent the magnitude of the diameter AB by $2r$, the magnitude of the segment (or *sagitta*) AC by x, we shall find



Example.

$$AD^2 = 2rx,$$

and therefore

$$AD = \pm \sqrt{2rx}.*$$

In this result, $\sqrt{2rx}$ represents indifferently the magnitude of the chord AD or Ad, but has no reference whatever to the position of one or the other of them with respect to AE: the introduction of the negative result therefore, is entirely due to the rules of Symbolical Algebra, and is, in no respect, relative to the problem proposed: and though it is quite true that if AD be assumed to be represented in magnitude and in direction by $\sqrt{2rx}$, then AD' which is equal and opposite to AD (and also any other equal line which is parallel to it) may be correctly represented by $-\sqrt{2rx}$, yet such an interpretation of the negative root, in the solution of the problem under consideration, would be a mere surplusage and altogether alien to the original assumptions which were made: in other words $\sqrt{2rx}$ may equally represent, under such circumstances, any line whatever which is equal to AD.

Ambiguities from roots of solution.

1048. The most common, however, of all the sources of ambiguity in the symbolical solution of problems, is due to the

* This example is also discussed by D'Alembert in the same Dissertation, referred to before.

introduction of extraneous factors which are required for the rationalization of the equations of condition, which the problem furnishes, and the consequent introduction of roots which will satisfy the *rationalized* but not the *primitive* equation; we have already illustrated this theory of *proper* roots of equations, as distinguished from *roots of solution*, in great detail, in a former Chapter (xx). The following is an example of a problem, whose solution leads to an equation of condition, of the peculiar class to which we are now referring.

"To determine a right-angled triangle the sum of whose two sides is 35, and of which the perpendicular let fall from the right angle upon the hypothenuse is equal to 12." Problem.

If we denote one of the two sides by x, the conditions of the problem will give us the equation

$$35x - x^2 = 12\sqrt{(2x^2 - 70x + 1225)},$$

which *rationalized* becomes

$$x^4 - 70x^3 + 937x^2 + 10080x - 176400 = 0.$$

This biquadratic equation is resolvible, by the processes given in Chapter xx, into the two quadratic equations

$$\left.\begin{array}{l} x^2 - 35x + 300 = 0 \\ x^2 - 35x - 588 = 0 \end{array}\right\}.$$

The roots of the first equation are 15 and 20, which satisfy the primitive equation of condition, and will be found to solve the problem: they express the two sides of the right-angled triangle.

The roots of the second equation, which are 47.4 and $-$ 12.4 nearly, satisfy the rationalizing factor

$$35x - x^2 + 12\sqrt{(2x^2 - 70x + 1225)} = 0$$

only, and not the primitive equation, and are therefore *roots of solution* only, and are not relative to the problem proposed.

1049. We must beware, however, lest we conclude too hastily, in all cases, where an irrational equation is rationalized, that some of the roots of the resulting equation are roots of solution merely, and not relative to the problem proposed: for it may happen that all the irrational factors of the rationalized equation The roots of a rationalizing factor are sometimes proper roots.

are equally furnished by the conditions of the problem, and that therefore all the roots which satisfy the several factors are equally relative to it: thus in the equation given in Art. 1044 the two irrational factors

$$a^2 - x^2 - c\sqrt{b^2 + x^2} = 0,$$

$$a^2 - x^2 + c\sqrt{b^2 + x^2} = 0,$$

of the final equation

$$x^4 - (2a^2 + c^2)\,x^2 + a^4 - b^2c^2 = 0,$$

are equally relative to the problem proposed, and their roots, therefore, equally satisfy the required conditions.

Ambiguities from unique or multiple values of expressions under a radical sign.

1050. Another source of ambiguity in Symbolical Algebra is referrible to the want of a uniform notation of radicals as well as of a uniform sense in which such notation is used.

Thus we denote the square root of $1 + x^2$ indifferently both by $\sqrt{1 + x^2}$, and by $(1 + x^2)^{\frac{1}{2}}$: the cube root of $1 - x^2$ either by $\sqrt[3]{(1 - x^2)}$, or by $(1 - x^3)^{\frac{1}{3}}$: the n^{th} root of $ax - x^2$ either by $\sqrt[n]{ax - x^2}$, or by $(ax - x^2)^{\frac{1}{n}}$: and similarly in other cases. No serious inconvenience could result from this variety of notation for identical quantities, inasmuch as they are immediately convertible into each other, provided it was used, in all cases, in precisely the same sense, or subject to the same limitations: but we sometimes assume $\sqrt{1 - x^2}$, or $\sqrt[4]{1 - x^2}$ to denote the multiple and sometimes the simple arithmetical root as determined by the principles of Arithmetical Algebra.

Suggestion for their removal.

1051. It would tend greatly to the clearness and precision of symbolical notation, if the radical sign $\sqrt{\ }$ or $\sqrt[n]{\ }$ was used strictly in the sense attached to it in Arithmetical Algebra, and the corresponding notation by indices to designate the multiple root of Symbolical Algebra: the equation

$$x^2 - 7x + \sqrt{(x^2 - 7x + 18)} - 24 = 0$$

would thus be confined to the designation of one of the two factors of the rationalized equation

$$(x^2 - 7x - 24)^2 - (x^2 - 7x + 18) = 0,$$

both of which would be equally represented by its symbolical form

$$x^2 - 7x + (x^2 - 7x + 18)^{\frac{1}{2}} - 24 = 0:$$

and it would be no sufficient objection to this limited use of the radical sign (which is exclusively used in Arithmetical Algebra, inasmuch as the use and theory of fractional indices is essentially symbolical) that it might be applied to expressions which were negative, and therefore not arithmetical: for expressions or equations in which such radical signs occur, can only admit of interpretation by being transferred to Symbolical Algebra.

1052. We have before noticed (Art. 960) the ambiguities which arise from the uncertainty which sometimes prevails, when the same specific radical presents itself in different terms of an equation or a series, whether the same specific value (when multiple values of it are admissible) which it is assumed to possess in one term is or is not *simultaneous* in all the others: if, however, we should adopt the limitations of notation which are proposed in the last Article, no ambiguity would occur in the equation

Ambiguities from assuming or not assuming the same specific value of the same radical expression in different terms of the same equation or series.

$$\sqrt{x^3} + 7\sqrt{x} = 5x + \frac{8}{\sqrt{x}},$$

or in the series

$$\frac{1}{\sqrt{(1+\sqrt{x})}} = 1 - \frac{\sqrt{x}}{2} + \frac{3x}{8} - \frac{5x\sqrt{x}}{16} + \frac{35x^2}{128} \dots,$$

where the specific radical $\sqrt{x}$ would retain throughout its single arithmetical meaning: but if the equation was

$$x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}} = \frac{1}{x^{\frac{1}{3}}} - 2,$$

where $x^{\frac{1}{2}}$ admits of double and $x^{\frac{1}{3}}$ of triple values, and where it is not assumed that the values either of $x^{\frac{1}{2}}$ or of $x^{\frac{1}{3}}$ are simultaneous on both sides of the equations, it would be equally capable of 36 different forms, and if rationalized would become an equation of 6 dimensions, all whose roots would be equally admissible as proper roots of the equation.*

1053. Such a range of uncertainty, however, would not be allowable in series or expressions derived from known operations in which the specific values of the radical terms involved in the generating expressions are, in all cases, assumed to be transmitted to the results to which they lead: thus if it was required to

Such ambiguities are not allowable in expressions derived from others by known operations.

* Appendix.

multiply $1+x^{\frac{1}{2}}$ into $1+x^{\frac{1}{3}}$, the specific values of $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ would be assumed to prevail in the product

$$1+x^{\frac{1}{2}}+x^{\frac{1}{3}}+x^{\frac{5}{6}}:$$

and in the developement of a series such as $(1-x^{\frac{1}{2}})^{-\frac{1}{3}}$ which gives

$$1+\frac{1}{3}x^{\frac{1}{2}}+\frac{2}{9}x+\frac{14}{81}x^{\frac{3}{2}}+\frac{35}{243}x^2\ldots$$

whatever be the specific value of $x^{\frac{1}{2}}$ which is assumed in the generating expression, the same is transferred to the series which is generated: in the absence of such an assumed community of values, the results of the operations of Symbolical Algebra would, in such cases, lead to inextricable confusion.

Ambiguities which arise when an expression and the ordinary form of its developement have not the same number of values.

1054. In the transition from a generating expression, more especially when it is deduced, in the first instance, for values of the symbols involved in it which are general in form but specific in value, and which are subsequently generalized "by the principle of the permanence of equivalent forms," or by means of some proposition to which the application of that principle leads, we sometimes adopt a form of the series whose value is unique when that of the expression in which it originates and to which it is assumed to be equivalent, is multiple: thus the value of the series

$$1+nx+n(n-1)\frac{x^2}{1.2}+n(n-1)(n-2)\frac{x^3}{1.2.3}+\ldots$$

which originates in $(1+x)^n$ is *unique* for all values of n: but inasmuch as the values of $(1+x)^n$ are multiple whenever n is fractional, the series in question, under such circumstances, ceases to be coextensive with, and therefore completely equivalent to, the expression from which it is derived: if however, we should call s the arithmetical value of the series

$$1+nx+n(n-1)\frac{x^2}{1.2}+n(n-1)(n-2)\frac{x^3}{1.2.3}+\ldots$$

we should find

$$(1+x)=1(1+x),$$

and

$$(1+x)^n=1^n(1+x)^n=1^n.s,$$

where 1^n is considered as (Art. 724) the *recipient* of the multiple values of $(1+x)^n$: and inasmuch as it appears (Art. 810) that

$$1^n=\cos 2nr\pi+\sqrt{-1}\sin 2nr\pi,$$

(where r may have any value in the series 0, 1, 2, 3 ...), we may form the equation

$$(1+x)^n = (\cos 2nr\pi + \sqrt{-1}\sin 2nr\pi)\,s,$$

which is true for all values of n.

We thus get

$$(1+x)^{\frac{1}{2}} = 1^{\frac{1}{2}}.\,s = (\cos r\pi + \sqrt{-1}\sin r\pi)\,s = \pm s,$$

$$(1+x)^{\frac{1}{3}} = 1^{\frac{1}{3}}.\,s = \left(\cos\frac{2r\pi}{3} + \sqrt{-1}\sin\frac{2r\pi}{3}\right)s$$

$$= s \text{ or } \left(\frac{-1+\sqrt{-3}}{2}\right)s, \text{ or } \left(\frac{-1-\sqrt{-3}}{2}\right)s,$$

and similarly in other cases.

The neglect of a sufficient attention to this peculiar source of ambiguity has led to many imperfect and erroneous generalizations, more particularly in the case of an extensive class of goniometrical series, some of which we shall now proceed to investigate: it is only within a very recent period that their correct forms were first exhibited by Poinsot.

1055. Let it be required to find a series for $(2\cos\theta)^m$ in terms of the cosines or sines of multiples of θ. Series for $(2\cos\theta)^m$ in terms of the sines or cosines of multiples of θ.

Let ρ represent the arithmetical value of $(2\cos\theta)^m$, and therefore

$$(2\cos\theta)^m = 1^m.\,\rho,$$

if $2\cos\theta$ be positive, and

$$(2\cos\theta)^m = (-1)^m\rho,$$

if $2\cos\theta$ be negative.

Again, since $2\cos\theta = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$ (Art. 926), we get

$$(2\cos\theta)^m = 1^m\,(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})^m,$$

$$= 1^m\left\{e^{m\theta\sqrt{-1}} + me^{(m-2)\theta\sqrt{-1}} + \frac{m(m-1)}{1.2}e^{(m-4)\theta\sqrt{-1}} + \ldots\right\}.$$

But $1^m = \cos 2mr\pi + \sqrt{-1}\sin 2mr\pi$,

$$e^{m\theta\sqrt{-1}} = \cos m\theta + \sqrt{-1}\sin m\theta,$$

$$e^{(m-2)\theta\sqrt{-1}} = \cos(m-2)\,\theta + \sqrt{-1}\sin(m-2)\,\theta,$$

$$e^{(m-4)\theta\sqrt{-1}} = \cos(m-4)\,\theta + \sqrt{-1}\sin(m-4)\,\theta,$$

....

Therefore

$$(\cos 2mr\pi + \sqrt{-1}\,\sin 2mr\pi)\,\rho$$

$$= (\cos 2mr\pi + \sqrt{-1}\,\sin 2mr\pi)\,\{\cos m\theta$$

$$+ m\cos(m-2)\,\theta + \frac{m(m-1)}{1\,.\,2}\cos(m-4)\,\theta + \ldots\}$$

$$+ \sqrt{-1}\,(\cos 2mr\pi + \sqrt{-1}\,\sin 2mr\pi)\,\{\sin m\theta$$

$$+ m\sin(m-2)\,\theta + \frac{m(m-1)}{1\,.\,2}\sin(m-2)\,\theta - \ldots\}:$$

but

$$(\cos 2mr\pi + \sqrt{-1}\,\sin 2mr\pi)\,(\cos m\theta + \sqrt{-1}\,\sin m\theta)$$

$$= \cos 2mr\pi\cos m\theta + \sqrt{-1}\,\sin 2mr\pi\cos m\theta$$

$$+ \sqrt{-1}\,\cos 2mr\pi\sin m\theta - \sin 2mr\pi\sin m\theta$$

$$= \cos m\,(2r\pi + \theta) + \sqrt{-1}\,\sin m\,(2r\pi + \theta).$$

We thus get

$$(\cos 2mr\pi + \sqrt{-1}\,\sin 2mr\pi)\,\rho,$$

$$= \cos m\,(2r\pi+\theta) + m\cos(m-2)\,(2r\pi+\theta)$$

$$+ \frac{m(m-1)}{1\,.\,2}\cos(m-4)\,(2r\pi+\theta) + \ldots$$

$$+ \sqrt{-1}\,\{\sin m\,(2r\pi+\theta) + m\sin(m-2)\,(2r\pi+\theta)$$

$$+ \frac{m(m-1)}{1\,.\,2}\sin(m-4)\,(2r\pi+\theta) + \ldots\}$$

$$= c_r + \sqrt{-1}\,s_r,$$

where c_r represents the series of cosines, and s_r the series of sines of the multiples of θ.

If we take the second case, in which $(2\cos\theta)^m = (-1)^m\rho$, we shall find, in a similar manner

$$\{\cos(2r+1)\,m\pi + \sqrt{-1}\,\sin(2r+1)\,m\pi\}\,\rho = c_r + \sqrt{-1}\,s_r,$$

where c_r and s_r represent the same series of cosines and sines of multiple angles as in the expression for $1^m\rho$, merely putting $(2r+1)\,\pi + \theta$ for $2r\pi + \theta$.

When m is a whole number.

1056. It remains to discuss the different forms which these series will assume for different values of m.

Let m be a whole number.

In this case, we may make r equal to zero in both members of the equation: or, in other words, we may replace c_r by c_0 and s_r by s_0,

If $2\cos\theta$ be positive and m even or odd $\rho = c_0$.

If $2\cos\theta$ be negative and m even $\rho = c_0$.

If $2\cos\theta$ be negative and m odd $\rho = -c_0$.

In all these cases, the value of ρ is expressed, therefore, by a series of cosines, the corresponding series of sines being equal to zero.

$$\text{The } (p+1)^{\text{th}} \text{ term of } c_0 = \frac{m(m-1)\ldots(m-p+1)}{1 \,.\, 2 \ldots p} \cos(m-2p)\theta$$

$$= t \cos(m-2p)\theta.$$

$$\text{The } (m-p)^{\text{th}} \text{ term of } c_0 = t\cos(m-2m+2p)\theta$$

$$= t\cos(m-2p)\theta.$$

The terms of the series for c_0 being, therefore, the same from the beginning to the end, it follows that

$$c_0 = 2\left\{\cos m\theta + m\cos(m-2)\theta + \frac{m(m-1)}{1 \,.\, 2}\cos(m-4)\theta + \ldots\right\}$$

continued to $\frac{m}{2}+1$ terms, when m is even, and to $\frac{m+1}{2}$ terms, when m is odd: the last term, in the first case, being

$$\frac{1\times 3\times 5\ldots(m-1)}{1\times 2\times 3\ldots \frac{m}{2}} \,.\, 2^{\frac{m}{2}-1},$$

which is a modified form of the half of the middle term of the binomial $(1+1)^m$ (Art. 489): and in the second case being

$$\frac{1\times 3\times 5\times\ldots m}{1\times 2\times 3\times\ldots \frac{m+1}{2}} \times 2^{\frac{m-1}{2}}\cos\theta.$$

1057. Let m be not a whole number. When m is not a whole number.

If $2\cos\theta$ be positive, we find

$$\rho = \frac{c_r}{\cos 2mr\pi} = \frac{s_r}{\sin 2mr\pi}.$$

If $2\cos\theta$ be negative, we find

$$\rho = \frac{c_r}{\cos m(2r+1)\pi} = \frac{s_r}{\sin m(2r+1)\pi}.$$

In both cases, therefore, ρ is expressible either by a series of cosines or of sines, unless $\cos 2mr\pi = 0$ or $\sin 2mr\pi = 0$ in one case, and $\cos m(2r+1)\pi = 0$ or $\sin m(2r+1)\pi = 0$ in the other.

In examining such cases, we may suppose m a rational fraction in its lowest terms of the form $\frac{p}{n}$, and also that r does not exceed $n-1$.

If p is odd, n divisible by 4 or *pariter par* (Art. 516, Note), and $r = \frac{n}{4}$ or $\frac{3n}{4}$, we get $\frac{2pr\pi}{n} = \frac{p\pi}{2}$ or $\frac{3p\pi}{2}$, and therefore $\cos\frac{2pr\pi}{n} = 0$.

In these cases ρ is expressible by a series of sines only, and $\rho = s_{\frac{n}{2}}$ or $-s_{\frac{3n}{2}}$, if p be of the form $4i+1$ or *pariter impar*, and $\rho = -s_{\frac{n}{2}}$ or $s_{\frac{3n}{2}}$, if p be of the form $4i+3$ or *impariter impar.*

Secondly, if $r = 0$, or if p be odd, n even and $r = \frac{n}{2}$, we find $\sin\frac{2pr\pi}{n} = 0$.

In these cases, ρ is expressible by a series of cosines only, and $\rho = c_0$ or $-c_{\frac{n}{2}}$.

Thirdly, if p be odd, n even and of the form $4i+2$ or *impariter par,* and $r = \frac{n-2}{4}$ or $\frac{3n-2}{4}$, we find $\cos\frac{p(2r+1)\pi}{n} = 0$.

In these cases, ρ is expressible by a series of sines only, and $\rho = -s_{\frac{3n-2}{4}}$ or $s_{\frac{3n-2}{4}}$, according as p is *pariter* or *impariter impar.*

Lastly, if n be odd and $r = \frac{n-1}{2}$, we find $\sin\frac{p(2r+1)\pi}{n} = 0$.

In these cases, ρ is expressible by a series of cosines only, and $\rho = c_{\frac{n-1}{2}}$ or $c_{\frac{n-1}{2}}$ according as p is even or odd.

1058. Again, let it be required to find a series for $(2\sin\theta)^m$, in terms of the sines or cosines of multiples of θ. Series for $(2\sin\theta)^m$.

Since $2\sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}$ (Art. 926), we get

$$(2\sqrt{-1}\sin\theta)^m = 1^m\{e^{m\theta\sqrt{-1}} - m\,e^{(m-2)\theta\sqrt{-1}} + \frac{m(m-1)}{1\,.\,2}e^{(m-4)\theta\sqrt{-1}} - \dots\}$$

$$= \cos m(2r\pi+\theta) - m\cos(m-2)(2r\pi+\theta) + \dots$$

$$\sqrt{-1}\{\sin m(2r\pi+\theta) - m\sin(m-2)(2r\pi+\theta) + \dots\}$$

$$= c_r + s_r\sqrt{-1},$$

if c_r be taken to represent the series of cosines, and s_r the series of sines.

If ρ denote the arithmetical value of $(2\sqrt{-1}\sin\theta)^m$, then, if $2\sin\theta$ be positive, we get

$$\{\cos m(2r+\tfrac{1}{2})\pi + \sqrt{-1}\sin m(2r+\tfrac{1}{2})\pi\}\rho = c_r + s_r\sqrt{-1}:$$

and if $2\sin\theta$ be negative

$$\{\cos m(2r+\tfrac{3}{2})\pi + \sqrt{-1}\sin m(2r+\tfrac{3}{2})\}\rho = c_r + s_r\sqrt{-1}.$$

1059. If m be a whole number and *even*, then When m is a whole number.

$$2^{m-1}(\sin\theta)^m = \pm\{\cos m\theta - m\cos(m-2)\theta + \dots$$

$$+ \frac{1\,.\,3\,.\,5\dots(m-1)}{1\,.\,2\dots\frac{m}{2}}\times 2^{\frac{m}{2}-1}\},$$

the sign + or − being used according as m is of the form $4i$ or $4i+2$.

If m be a whole number and *odd*, then

$$2^{m-1}(\sin\theta)^m = \pm\{\sin m\theta - m\sin(m-2)\theta + \dots \text{ to } \frac{m+1}{2} \text{ terms}\},$$

the sign + or − being used, according as m is of the form $4i+1$ or $4i+3$.

1060. If m be not a whole number and $2\sin\theta$ positive, then When m is not a whole number.

$$\rho = \frac{c_r}{\cos m(2r+\tfrac{1}{2})\pi} = \frac{s_r}{\sin m(2r+\tfrac{1}{2})\pi},$$

or its value is expressible either by a series of cosines or sines of multiple angles, except under the following circumstances.

If $m = \frac{p}{n}$ (in its lowest terms), p and n being odd numbers, then

$$\rho = s_{\frac{n-1}{4}} \text{ or } - s_{\frac{n-1}{4}},$$

according as p is of the form $4i+1$ or $4i+3$.

If p be even and n of the form $4i+1$, then

$$\rho = c_{\frac{n-1}{4}} \text{ or } - c_{\frac{n-1}{4}},$$

according as p is of the form $4i$ or $4i+2$.

If m be not a whole number and $2 \sin \theta$ negative, then

$$\rho = \frac{c_r}{\cos m\,(2r + \frac{3}{2})\,\pi} = \frac{s_r}{\sin m\,(2r + \frac{3}{2})\,\pi}.$$

its value being expressible, either by a series of sines or of cosines, except under the following circumstances.

If $m = \frac{p}{n}$ (in its lowest terms), p and n being odd numbers, then

$$\rho = \pm s_{\frac{n+1}{4}} \text{ or } \pm s_{\frac{3n+1}{4}},$$

when n is of the form $4n+3$, the sign $+$ or $-$ being used, according as p is of the form $4i+1$ or $4i+3$ in the first case, or the contrary in the second.

If p be even and n of the form $4i+3$, then

$$\rho = \pm c_{\frac{n+1}{4}} \text{ or } \mp c_{\frac{3n+1}{4}},$$

the $+$ or $-$ sign being used, according as p is of the form $4i$ or $4i+2$ in the first case, or the contrary in the second.

We have been thus minute and critical in the deduction of all the separate cases which this problem comprehends, not only on account of the intrinsic importance of the problem itself, but likewise as affording a very instructive example of the proper mode of discussing and interpreting a formula when it is expressed in very general terms: for it will very generally be found, that the more comprehensive is the form in which a problem is stated

and investigated, the more remote and difficult its application will be to the particular cases which it includes.

A converse problem to the one which we have just been considering, would require us to assign a series for $\cos m\theta$ and $\sin m\theta$ in terms of the sines and cosines of θ: the investigation however of such series, which branch out into a great variety of forms and cases, cannot easily be effected without the aid of principles and processes which will be given in a subsequent volume of this work. Series for $\cos m\theta$ and $\sin m\theta$, in terms of $\cos\theta$ or $\sin\theta$.

1061. The same principles will likewise find their application in the general theory of symbolical as distinguished from arithmetical logarithms: if we assume ρ to represent the arithmetical logarithm of a or of its powers, we may extend our enquiries to determine the most general symbolical forms of the logarithms of $(1\times a)^m$, of $(-1\times a)^m$, or of $-1\times(1\times a)^m$, or, in other words, we may suppose a or a^m to be affected by any sign, which is recognized in Symbolical Algebra, and their logarithms to be required, assuming e to be the common base to which they are referred. Theory of symbolical as distinguished from Arithmetical logarithms.

Since it has been shewn in Art. 927, that The Napierian logarithms of 1^m and $(-1)^m$.

$$1^m = \cos 2mr\pi + \sqrt{-1}\,\sin 2mr\pi = e^{2mr\pi\sqrt{-1}},$$

and

$$(-1)^m = \cos m(2r+1)\pi + \sqrt{-1}\,\sin m(2r+1)\pi = e^{m(2r+1)\pi\sqrt{-1}},$$

it will follow, in conformity with the definition of Napierian logarithms (Art. 901) that

$$\log 1^m = 2mr\sqrt{-1}.$$

$$\log(-1)^m = m(2r+1)\pi\sqrt{-1}.$$

The logarithm of 1^m is *zero*, when $m=0$ or $r=0$: the logarithm of $(-1)^m$ can only become *zero* when $m=0$ and therefore $(-1)^0=1$: the other logarithms are all imaginary, and are unlimited in number: it is the logarithm of 1^0 or $(-1)^0$ only, which is essentially zero and which admits of no other value.

1062. If m be a fraction in its lowest terms, with an *even denominator* of the form $\frac{p}{2n}$, there is one value of 1^m which is equal to -1, corresponding to $r=n$: in this case we find Case in which one value of the logarithm of 1^m coincides with one value of a logarithm of -1.

$$\log 1^m = 2\cdot\frac{p}{2n}\cdot n\pi\sqrt{-1} = p\pi\sqrt{-1},$$

where p is an odd number, a result which coincides with one of the values of $\log(-1)^m$, when $m=1$ and $r=\frac{p-1}{2}$.

Symbolical logarithms of a^m.

1063. Inasmuch as

$$a^m = (1 \times a)^m = 1^m \times a^m,$$

we find

$$\log a^m = \log 1^m + \log a^m,$$

$$= 2mr\pi\sqrt{-1} + \rho,$$

where ρ is the arithmetical logarithm of a^m.

When we seek for the general logarithms of a^m, we presume that a is viewed as a symbolical and not as an arithmetical quantity, and we replace it by $1 \times a$, where 1 is made the recipient of the affections of a, whilst the other factor a, is considered as an arithmetical magnitude merely: this is a distinction of fundamental importance, both in this and other theories, and which our ordinary notation is not competent to express, a defect which is a constant and very embarrassing source of ambiguity*.

The symbolical logarithms of $(-a)^m$.

1064. Again, since

$$(-a)^m = (-1)^m \times a^m,$$

we get

$$\log(-a)^m = \log(-1)^m + \log a^m,$$

$$= m(2r+1)\pi\sqrt{-1} + \rho.$$

* Mr D. F. Gregory, in a very able memoir "On the Impossible Logarithms of Quantities," which is given in the first volume of the Cambridge Mathematical Journal, has proposed to distinguish these arithmetical and symbolical values, by a and by $+a$, where the sign $+$ takes the place of 1, in the notation adopted in the text, as the recipient of the affections of a, and where $+^m$ is considered as equivalent to 1^m: such a use of the sign $+$, as the subject of symbolical operations, is opposed to the ordinary conventions of notation, and is calculated to keep out of sight the peculiar symbolical properties of the various roots of 1, by which all our signs of affection are symbolized: it is certainly no objection to its use, as urged by Mr Gregory, that its tendency is to recall arithmetical ideas.

I cannot refer to this memoir of Mr Gregory, the inheritor of the name and honours of a family singularly illustrious in the history of the sciences, without expressing the deep sense which I feel, in common with all who knew him, of the loss which the mathematical and philosophical world has sustained by his premature death: his memoirs were remarkable for the large and original views which they take of the principles of mathematical reasoning, and gave ample promise of the valuable results which could hardly have failed to have followed from the full developement of his powers.

If $m=2$, we find

The logarithms of $(-a)^2$ and a^2, are not identical.

$$\log(-a)^2 = (4r+2)\pi\sqrt{-1} + 2\log a.$$

It appears, from the last Article, that

$$\log a^2 = 2r\pi\sqrt{-1} + 2\log a.$$

It follows, therefore, that, though

$$(-a)^2 = a^2 *$$

and though the logarithms of $(-a)^2$ are always included amongst those of a^2, the converse proposition is not true: it consequently appears that we are not authorized in inferring, as has sometimes been done, the identity of the logarithms of $(-a)^2$ and of a^2, from the identity of the symbolical results to which they lead when the signs of affection or their recipients are suppressed.

1065. Similarly, since

The symbolical logarithms of $-a^m$.

$$-a^m = (-1)\times 1^m \times a^m,$$

we find

$$\begin{aligned}\log - a^m &= \log(-1) + \log 1^m + \log a^m \\ &= (2r+1)\pi\sqrt{-1} + 2mr'\pi\sqrt{-1} + \rho \\ &= (2r+2mr'+1)\pi\sqrt{-1} + \rho.\end{aligned}$$

This is the most general form, in which the logarithm of $-a^m$ can be exhibited, and it remains to consider, whether it admits, in any case, of an arithmetical value.

The logarithm of a negative quantity

* Mr Gregory, in the memoir to which I have before referred, has concluded from the equation

$$(-1)^2 = 1,$$

or, as expressed in his notation

$$(-)^2 = +$$

$$\text{that } - = +^{\frac{1}{2}},$$

and he infers from thence that $+^{\frac{2n+1}{2}}$ is the general representation of the sign $-$: but it might, with equal propriety have been taken, as the general representation of the sign $+$; for the equation

$$+^2 = +$$

is equally true with the equation

$$(-)^2 = +.$$

can, in no cases, be possible or equal to that of a positive quantity.

If we suppose, as in Art. 1062, m, in its lowest terms, to be of the form $\frac{p}{2n}$, and if we make $r'=n$, and $2r+1=-p$, then

$$(2r+2mr'+1)=0,$$

and there would apparently be one value of the logarithm of $-a^m$, which is possible and equal to ρ or to the arithmetical value of $\log a^m$.

A very little consideration, however, will shew that this case is not admissible*.

For one of the values of $1^{\frac{p}{2n}}$ is, as we have already shewn, (Art. 1062) equal to -1, which is that which corresponds to $r'=n$: and inasmuch as

$$-1=(-1)^{-2r-1}=(-1)^{-p},$$

it will follow, that, under such circumstances,

$$-a^m=(-1)\ (1)^m a^m=(-1)\ (1)^{\frac{p}{2n}} a^m$$

$$=(-1)\ (-1)\, a^{\frac{p}{2n}}=a^{\frac{p}{2n}},$$

a form which is necessarily excluded, inasmuch as it is assumed that $(-1)\ (1\times a)^m$ is equal to $-a^m$ and not a^m: we may conclude, therefore, generally that there is no possible logarithm of a negative quantity.

Difficulty of avoiding ambiguities and errors in the theory of symbolical logarithms.

1066. The ambiguity which necessarily results from the identical representation of arithmetical and symbolical quantities, and the essential difference of the form of their logarithms, makes a vigilant attention to the most delicate shades of difference between them absolutely necessary, in order to avoid defective or excessive generalizations: thus some writers have considered the base of the system of logarithms as equally capable of multiple values with the quantities whose logarithms are required, not sufficiently considering that the system of logarithms itself is altered by every change of its base: thus, if we suppose

$$y=a^x,$$

they replace y by $1\times y$, and a^x by $1^x\times a^x$, assuming a^x in $1^x\times a^x$ to be arithmetical: we thus find

* The existence of a possible logarithm of a negative number was asserted in my first work on Algebra, page 569, and the mistake was pointed out by Mr Gregory, in the memoir to which I have before referred.

$$\log 1 + \log y = \log 1^x + \log a^x,$$

$$2r\pi\sqrt{-1} + \log y = x\,(2r'\pi\sqrt{-1}) + x\,;$$

and therefore

$$x = \frac{\log y + 2r\pi\sqrt{-1}}{1 + 2r'\pi\sqrt{-1}},$$

and if $y = 1$, we get x or the logarithm of 1

$$= \frac{2r\pi\sqrt{-1}}{1 + 2r'\pi\sqrt{-1}},$$

and not $2r\pi\sqrt{-1}$, as determined above, Art. 1061.

APPENDIX.

On the limits of the values of the symbols in Arithmetic, Arithmetical and Symbolical Algebra.

Page 12, Art. 558. In Arithmetical Algebra, *zero* and infinity are the extreme limits of the values of the symbols which we employ, though the circumstances of their usage will very generally confine them within a much less extensive range.

Thus, in the expression $a-b$, a may have every value between infinity and b, whilst b may have every value between a and 0: the expression $a-b$ itself may have every value between 0 and infinity.

But in Symbolical Algebra, a, b, and $a-b$ may severally have every value between positive and negative infinity, zero being included in their number: and if, as in the case of lines drawn in opposite directions, the positive and negative values equally admit of interpretation and are consequently equally *real*, it will follow that their symbolical is coextensive with their geometrical representation.

It may be further remarked that Arithmetical and Symbolical Algebra are equally competent to represent continuous magnitude between the limits of the values of their symbols, or of the expressions which they form.

Magnitudes represented by the symbols of Arithmetic are discontinuous.

The magnitudes represented by the symbols of Arithmetic and by the expressions, whether fractional or decimal, which they form, are essentially discontinuous, being incompetent to express the continuous values which are included between their successive units: and it is only by the indefinite subdivision of the primary units, whatever they may be, that we can approximate, in this science, to the representation of continuous magnitude.

Magnitudes represented by the symbols of Arithmetical and Symbolical Algebra are continuous.

On the contrary, the symbols of Arithmetical or Symbolical Algebra are incompetent to represent, like those of Arithmetic, discontinuous magnitudes: for whilst in Arithmetic, magnitudes are represented *at* their limits only, however numerous they may be, in the other science, they are represented *between* their limits only, whatever those limits may be.

It is, for this reason, that the symbols of Algebra are not competent to express generally the properties of numbers, which are essentially discontinuous in their nature: and it is owing to this cause that the theory of numbers, properly so called, as distinguished from the theory of numerical operations, being not reducible to the general representations of symbolical language, has made no advancement which is comparable to that of the kindred science of Algebra.

Page 52, Art. 622. We have assumed the rules for the addition and subtraction of numerical fractions as the basis of the corresponding rules for those operations in Symbolical Algebra: but a further consideration of the principles upon which those rules are deduced (Art. 127 and 128) would rather lead to the conclusion that the interpretation of the meaning of numerical fractions which is there assumed, is not strictly deducible from the meaning attached to the quotient of the division of one number by another, with which it is required to be coincident.

Assuming, therefore, that the only property which the definitions of the fundamental operations of arithmetic would assign to numerical fractions is, that their values are not altered when their numerators and denominators are multiplied or divided by the same number, we will consider the case of two numerical fractions $\frac{m}{p}$ and $\frac{n}{p}$, which are to be added together; their sum will be represented by $\frac{m}{p}+\frac{n}{p}$, and if we further suppose that amongst the different values which may be given to m and n, there are some which are multiples of p, so that $\frac{m}{p}=\frac{rp}{p}=r$ and $\frac{n}{p}=\frac{sp}{p}=s$, then, under such circumstances, we should find

$$\frac{m}{p}+\frac{n}{p}=r+s=\frac{(r+s)\,p}{p}$$

$$=\frac{rp+sp}{p}=\frac{m+n}{p}:$$

it appears, therefore, that when $m=rp$ and $n=sp$, we get

$$\frac{m}{p}+\frac{n}{p}=\frac{m+n}{p}:$$

and inasmuch as this formula or expression represents the sum

of the fractions $\frac{m}{p}$ and $\frac{n}{p}$ when the symbols involved in it are general in their form, but specific in their value, it will continue to represent their sum when the symbols in it are general both in their form and value: in a similar manner, it may be shewn that

$$\frac{m}{p}-\frac{n}{p}=\frac{m-n}{p}.$$

Considering therefore such fractions, in the first instance, as expressing the quotient of the division of the numerator by the denominator, we should proceed to interpret their meaning, as in the case of the results of all other symbolical operations, with reference to the symbolical conditions which they are required to satisfy; we shall thus be readily enabled to arrive at the interpretation of their meaning, which we have made the foundation of the conclusions in the text.

The principle of the permanence of equivalent forms is the most general expression of the connection which exists between Arithmetical and Symbolical Algebra.

Art. 631. "The principle of the permanence of equivalent forms" is that which expresses, in the most general terms, the nature of the connection between Arithmetical and Symbolical Algebra.

All the conclusions of Arithmetical Algebra are considered to be the necessary results of the defined operations of addition, subtraction, multiplication and division, involution and evolution, when applied to numbers or quantities whose relations are fully understood: and such conclusions when represented through the medium of symbols which are general in form but specific in value, or by rules which are general in their form, though applied to quantities which are specific in their value, are assumed to be true likewise when the symbols which represent such magnitudes are equally general in their form and representation.

In Arithmetical Algebra it is the definition of the operation (whether expressed or understood) which determines the result, and also the rule for obtaining it: in Symbolical Algebra it is the rule which determines the meaning of the operation, such rule being determined by the principles of Arithmetical Algebra, when the symbols, though general in their form, are yet so specific in their value, as to come under the operation of its definitions: the rules of operation are the same in Arithmetical and Symbolical Algebra, and therefore the results are the same as far as they proceed in common: but it is at the point of transition from Arithmetical to Symbolical Algebra, when the

symbols or the conditions of their usage, cease to be arithmetical, that the meaning of the operations must be determined, not by definition, but interpretation: and such interpretations must vary with every change in the circumstances of their application.

The results of Arithmetical Algebra may be said to exist by *necessity*, as consequences of the definitions: and those definitions, whether expressed or understood, (for they are never formally enunciated) may be considered as derived immediately from the relations of numbers, and as consequently involving nothing which is arbitrary or variable in our conceptions of their nature or essence: they may be said, therefore, to possess in an eminent degree, the character of mathematical necessity.

The results of Arithmetical Algebra are necessary, those of Symbolical as far as they are not common to Arithmetical, Algebra, are conventional.

The case is very different, however, with the results of Symbolical, as far as they are not common likewise to Arithmetical, Algebra: inasmuch as they may be said to exist by convention only, for the rules for forming them are not proved as consequences of definitions, but are borrowed or adapted from a kindred science: and it is only when specific values are assigned to the symbols, that their relations or properties can become the subject of our reasonings, with a view to their interpretation in those cases and in those cases only, where the requisite correspondence, between the symbols and the quantities which they are assumed to represent, can be shewn to exist.

Again, such interpretations must not be confined to the meaning of the operations performed merely, but must extend likewise to the nature of the connection which exists between the operation and its result: the sign =, which is universally used for this purpose, means arithmetical equality in Arithmetical Algebra, when placed between the primitive expression and the result of the operation which it involves, whether the result, to which it leads, presents itself under a finite or indefinite form: but in Symbolical Algebra, in cases which are not likewise common to Arithmetical Algebra and in which the operation which produces the result requires interpretation, the sign =, in common with the expressions which it connects, must necessarily be included in it: its most comprehensive meaning will be that the expression which exists on one side of it is the result of an *operation* (using this term in its largest sense, Art. 632) which is indicated on the other side of it and not performed: this view of its general meaning will include as a consequence, arith-

The interpretation of operations must be extended to the sign = which connects the primitive expression and the result derived from it.

metical equality or algebraical equivalence, according as either one or the other of them may be shewn to exist.

Expressions may be algebraically equivalent which are not equal.

The phrase algebraical equivalence, as distinct from algebraical identity or arithmetical equality, would be applied in the case of expressions which, though not possessing either of these characters, are capable of reproducing or representing the symbolical properties of the expressions from which they are derived: thus in the case of the series which result from the developement of binomials such as $(1+x)^m$ and $(1+x)^n$, by the general theorem considered in Chap. XXI, we find

$$(1+x)^m = 1^m\{1 + mx + m(m-1)\frac{x^2}{1\,.\,2} + \dots\} \qquad (1),$$

$$(1+x)^n = 1^n\{1 + nx + n(n-1)\frac{x^2}{1\,.\,2} + \dots\} \qquad (2),$$

and it appears that for all values of x, m, and n, we have

$$(1+x)^m \times (1+x)^n = (1+x)^{m+n},$$

and likewise that the product of the series (1) and (2), if they are multiplied together, term by term, according to the ordinary rule for that purpose, will be the series

$$1^{m+n}\{1 + (m+n)x + (m+n)(m+n-1)\frac{x^2}{1\,.\,2} + \&c. \dots\},$$

which arises from replacing m in the series (1), or n in the series (2), by $m+n$: it thus appears that the products of the binomials $(1+x)^m$ and $(1+x)^n$, and of the series (1) and (2), are equally formed by replacing m or n by $m+n$, in the binomials and the series corresponding to them, and in this sense, and to this extent, they are said to be equivalent to each other: but if m or n be likewise rational numbers and x less than 1, then the binomial $(1+x)^m$, and the series (1) corresponding to it, may be considered not merely as *algebraically equivalent*, in the sense which we have attached to the term, but likewise as *arithmetically equal* to each other.

Misapplication of the term false to diverging series.

It has become a common practice with many distinguished analysts to denounce all diverging series as either *false* or *insecure*, or, in other words, as not capable of replacing the expression in which they originate in any algebraical operation, without leading to erroneous results: it would, however, be more in accordance with large and comprehensive views of Symbolical Algebra and its operations, if it should be said that diverging

series are *not arithmetical*, and therefore incapable of arithmetical computation by the aggregation of their terms: that inasmuch as they *rarely, if ever*, originate in expressions which are arithmetical both in their arrangement and value, it would be more correct to term them *false* under such circumstances, if they gave arithmetical values of expressions which were not themselves arithmetical: and further, that an algebraical equivalence may exist between an expression and its developement, when they are not arithmetically equal: and likewise that whenever general processes of Arithmetical Algebra may enable us to determine the expression which generates a series, when the symbols which it involves, though general in their form, are so modified in value as to produce a series which is arithmetically equal to it, they will equally enable us to assign it, when those symbols are general both in their form and value. (Art. 958, Note).

Page 106, Art. 680. The existence and form of the series for $(1+x)^n$, is proved, when n is a whole number, by considerations derived from the fundamental operations of Arithmetic and Arithmetical Algebra: and the same series, exhibited under an interminable form, is extended to all values of the index, by the principle of the permanence of equivalent forms.

The primary assumption of indices in Arithmetical Algebra leads directly to the propositions

$$(1+x)^m \times (1+x)^n = (1+x)^{m+n},$$

$$\{(1+x)^m\}^n = (1+x)^{mn},$$

where m and n are whole numbers: and we derive from the same assumption of indices, the extension of the same conclusions to the series which are deduced from them, when m and n are whole numbers, or in other words, we are thus enabled to conclude that the product of the series for $(1+x)^m$ into the series for $(1+x)^n$ will be the series for $(1+x)^{m+n}$, or, in other words, their product will be that series which arises from putting $m+n$ in the place of m in one series, or of n in the other: and also that the series for the n^{th} power of $(1+x)^m$, which is $(1+x)^{mn}$, is formed by putting mn in the place of m in the series for $(1+x)^m$. It is the principle of the permanence of equivalent forms, as we have shewn before (Appendix, p. 450), which enables us to extend these conclusions to all values of m and n.

Euler had drawn the same conclusion, nearly in the same manner, in his celebrated proof of the series for $(1+x)^m$ *, though he at the same time denied the universal application of a principle equivalent to that of the permanence of equivalent forms, which alone could make it valid: he produced, as a striking exception to its truth, the very remarkable series

$$\frac{1-a^m}{1-a}+\frac{(1-a^m)(1-a^{m-1})}{1-a^2}+\frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3}+\ldots$$

whose sum is m, when m is a whole number, but not so for other values.

A little consideration, however, will be sufficient to shew that the principle of the permanence of equivalent forms is not applicable to such a case: for if m be a whole number, as in Arithmetical Algebra, the connection between m and its equivalent series in the identical equation

$$m=\frac{1-a^m}{1-a}+\frac{(1-a^m)(1-a^{m-1})}{1-a^2}+\frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3}+\ldots$$

is not given, or, in other words, there is no statement or definition of the operation, by which we pass from m, on one side of the sign $=$, to a series under the specified form on the other, and there is consequently no basis for the extension of the conclusion to all values of the symbols, either by the principle of the permanence of equivalent forms or by any other: it is only when the results, which are general in form, but specific in value, are derived by processes which are definable and recognized, that they become the proper subjects for the application of this principle.

Page 120, Art. 695. The fundamental operations of Symbolical Algebra conduct us necessarily to expressions such as $+a$ or a, and $-a$, $(+a)^{\frac{1}{n}}$ or $a^{\frac{1}{n}}$, and $(-a)^{\frac{1}{n}}$, which are easily shewn to be symbolically equivalent to $+1\times a$ or $1\times a$ and $-1\times a$, $(+1)^{\frac{1}{n}}\times a^{\frac{1}{n}}$ or $1^{\frac{1}{n}}\times a^{\frac{1}{n}}$, and $(-1)^{\frac{1}{n}}\times a^{\frac{1}{n}}$, where a and $a^{\frac{1}{n}}$ are used as in Arithmetical Algebra: it would thus appear that 1 and -1 may be very conveniently used as the recipients (Art. 724) of the known and recognized signs of affection of Algebra: for all modes of representation in this science are considered to be

* Acta Petropol. for 1774.

equivalent to each other, which lead, in conformity with its rules, to identical symbolical forms*.

This replacement of the signs of affection in Symbolical Algebra, by the roots of 1, considered as their *recipients* and representatives, will enable us to consider all expressions in Algebra as composed of two factors, one being the appropriate root of 1 or -1, and the other an arithmetical magnitude, or, in other words, such a magnitude as is recognized in Arithmetical Algebra: and what is more important, it will further enable us to determine the proper symbolical forms of such signs of affection, and thus to reduce the expressions, which involve them, to their most simple equivalent forms: it is this circumstance which gives such peculiar importance to the theory of the roots of 1, and which connects their properties and their symbolical determination so essentially with the progress of Algebra.

Page 343, Art. 979. The subject of the roots of solution of equations which are formed in the solution of Geometrical problems has been discussed at some length in Art. 1044, and those which follow it. It is there shewn, that it by no means follows, that negative roots, though interpretation may make them real, will be therefore relative to the problem proposed: and what

* Mr Gregory, in an article above referred to, p. 442, distinguished $+a$ and $-a$ from each other, the former as belonging to Symbolical Algebra and considered as opposed to $-a$, and the other as a simple unaffected symbol in Arithmetical Algebra: such a distinction, however, would rarely enable us to remove ambiguities, if it could be preserved, and it is farther opposed to the very important and fundamental rule, which allows us to suppress the sign $+$ when it is connected with a symbol which has no other symbol before it: it is contrary to all just views of the relations of Arithmetical and Symbolical Algebra, to retain any distinctions between them which the ordinary rules of operation would lead us to suppress.

Mr Gregory further proposes to make the primary signs $+$ and $-$ themselves the subjects of operations, and to use $(+)^{\frac{1}{n}}$ instead of $1^{\frac{1}{n}}$, and $(-)^{\frac{1}{n}}$ instead of $(-1)^{\frac{1}{n}}$: considering the use of 1 and -1, as the recipients of signs of affection, as objectionable, in consequence of their tending to recal arithmetical notions, when the circumstances, in which such signs originate, present themselves exclusively in Symbolical Algebra, and may, therefore, according to his views, be altogether independent of arithmetic: but there may be a derivative though not an immediate dependance of one of these sciences upon the other, and I believe that no views of the nature of Symbolical Algebra can be correct or philosophical which made the selection of its rules of combination arbitrary and independent of arithmetic.

is more, that even positive roots may, in some cases, fail to answer the required conditions.

Page 357, Art. 986. The entire consequences deducible from the equation of condition

$$\sqrt{tt't''} = r$$

are not correctly stated in the text, inasmuch as no notice is taken of the case in which r is negative as well as positive: it should be added, that if r be negative, then all the values of $\sqrt{t}$, $\sqrt{t'}$, $\sqrt{t''}$ are negative, or two of them are positive: we should thus find, that when r is positive, we have

$$\alpha = \tfrac{1}{2}(\sqrt{t} + \sqrt{t'} + \sqrt{t''}),$$
$$\beta = \tfrac{1}{2}(\sqrt{t} - \sqrt{t'} - \sqrt{t''}),$$
$$\gamma = \tfrac{1}{2}(\sqrt{t'} - \sqrt{t} - \sqrt{t''}),$$
$$\delta = \tfrac{1}{2}(\sqrt{t''} - \sqrt{t} - \sqrt{t'}):$$

and if r be negative, we have

$$\alpha = \tfrac{1}{2}(-\sqrt{t} - \sqrt{t'} - \sqrt{t''}),$$
$$\beta = \tfrac{1}{2}(\sqrt{t} + \sqrt{t'} - \sqrt{t''}),$$
$$\gamma = \tfrac{1}{2}(\sqrt{t} + \sqrt{t''} - \sqrt{t'}),$$
$$\delta = \tfrac{1}{2}(\sqrt{t'} + \sqrt{t''} - \sqrt{t}).$$

The theory of this solution, as applicable to the complete equation

$$x^4 - px^3 - qx^2 - rx - s = 0$$

is still more completely developed in Art. 1028.

Page 360, Art. 989. The argument of Abel entitled "Demonstration de l' impossibilitè de la resolution algebrique des equations generales qui possent le quatrième degrè," is given in the first volume of his works* and has been revised and amplified by Sir William Hamilton in the 18th volume of the Transactions of the Royal Irish Academy: and though no subject could pass through the hands of those great analysts without retaining the impression of their extraordinary sagacity and power, yet it may easily be conceived that even they have failed to make investigations of a character so refined and difficult

* Published by Professor Holmboe at Christiania, 1839.

intelligible to a reader whose mind has not been rendered long familiar with the highest generalizations of symbolical language.

There is only one remark which I would venture to make on the subject of this demonstration: its authors would appear to have omitted to notice the effect of equations of condition in limiting the multiple values of the final expression for the roots, a consideration which enters essentially into all speculations on the general solution of equations.

Page 404, Art. 1089. Attempts have been made by Meyer Hirsh of Berlin in the first instance, and subsequently by Professor Badano of Genoa, to discover *cyclical* periods amongst the roots of this reducing equation and thus to apply the methods, which are shewn in the Articles which follow, to be successful with binomial equations, to the solution of an equation of the fifth degree: the error of the first attempt was subsequently discovered and acknowledged by its author, a mathematician of no ordinary attainments and merit: that of the second, which presents itself in a very plausible form, has been made the subject of a singularly elegant investigation by Sir William Hamilton in the 19th volume of the Transactions of the Royal Irish Academy.

Page 365, Art. 996. It is to the form

$$u^5 - 5su^3 + 5s^2u - a = 0,$$

that the resolution of a general equation of the fifth degree has been attempted to be reduced by Mr Jerrard, in some very remarkable researches in which he has succeeded in shewing that this great problem may be reduced to the resolution of an equation under the very simple form

$$x^5 + x + e = 0,$$

where e involves $\sqrt{-1}$: their failure in advancing beyond this point, has been shewn by Sir William Hamilton in an "Inquiry into the validity of a method recently proposed by George B. Jerrard, Esq., for transforming and resolving equations of elevated degrees," which appears in the sixth Report of the British Association for the advancement of Science.

THE END.